Lineaire tijdsonafhankelijke systemen en hun mathematisch model

Samenvatting

Gilles Callebaut

Hoofdstuk 1 : Lineaire tijdsonafh. systemen en hun mathematisch model

Het mathematisch model

Voor een LTO : $B_n \frac{d^n y}{dt^n} + \dots + B_0 y = A_m \frac{d^m u}{dt^m} + \dots + A_0 u$

Lineariteit:

↳ voldoen ah superpositie-principe

$$
\begin{array}{l}
u_1(t) \to y_1(t) \\
+ \; u_2(t) \to y_2(t) \\
\hline
u_1 + u_2 \to y_1 + y_2
\end{array}
$$

Tijdsafhankelijk:

$u(t) \to y(t)$

dan ↳ $u(t-\tau) \to y(t-\tau)$

Causaliteit:

op een bepaald tijdstip t

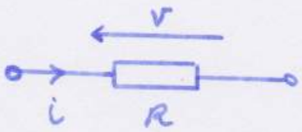
↳ uitgangssignaal enkel afhankelijk vh op dat moment ingangssignaal

↑

kan de toekomst n̄t voorspellen

Eenvoudige elektrische L.T.O. - Systemen

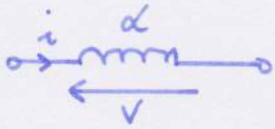
Weerstand



$$v = Ri \qquad\qquad i = \frac{v}{R}$$

$dW = v\,dq = Ri^2\,dt$ ← warmte door het Joule-effect. (niet-reversibel)

Spoel

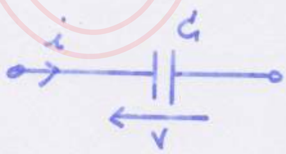


$$v = \alpha \frac{di}{dt} \quad \rightarrow \quad i = \frac{1}{\alpha}\int_0^t v\,dt + i_0$$

$$dw = v\,dq = \alpha \frac{di}{dt} . dq$$

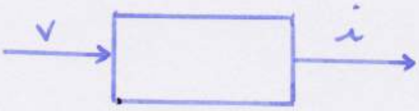
$$= \alpha\, d\left(\frac{i^2}{2}\right)$$ ← magn. veld (reversibel)

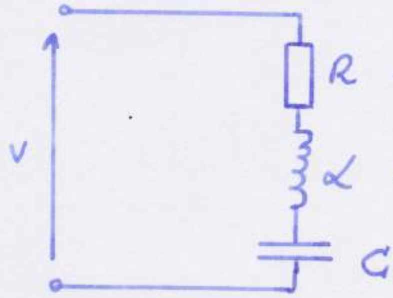
Condensator



$$v = v_0 + \frac{1}{C}\int_0^t i\,dt \quad \leftarrow \quad i = C\frac{dv}{dt}$$

$dW = v\,dq = \frac{1}{C} dq^2 = d\left(\frac{q^2}{2C}\right)$ ← elektr. veld (reversibel)





$\to v_R = Ri$

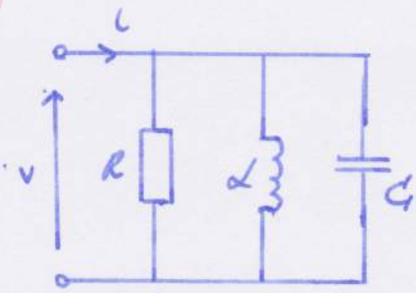
$\to V_{\mathcal{L}} = \mathcal{L}\frac{di}{dt}$

$\to i = C\frac{dv_c}{dt}$

$$\mathcal{L}\frac{di}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dv}{dt}$$

$$i = \frac{dq}{dt}$$

$$\mathcal{L}\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{1}{C}q = v$$



$\to i = C\frac{dv}{dt}$

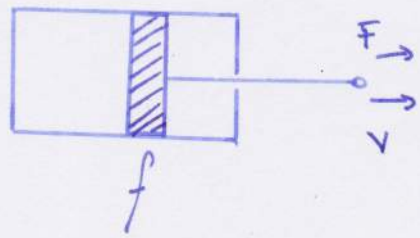
$i_R = \frac{V}{R}$ $\quad V = \mathcal{L}\frac{di_L}{t}$

$$C\frac{d^2v}{dt} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{\mathcal{L}}v = \frac{di}{dt}$$

Eenvoudige mechanische L.T.O. - systemen

translatie

Wrijvingsweerstand

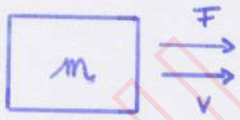


$$v = \frac{1}{f} F \longleftarrow F = f v$$

$$dW = F\,dx$$
$$= f v\,dx$$
$$= f v^2\,dt$$

← wrijvingsweerstand (warmte)

Massa

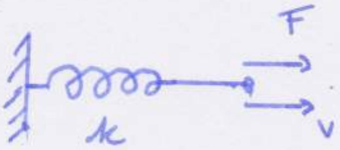


$$v(t) = v(0) + \int_0 F\,dt \longleftarrow F = m \frac{dv}{dt}$$

$$dW = m \frac{dv}{dt}\,dx$$
$$= d\left(\frac{mv^2}{2}\right)$$

← kinetische energie

Veer



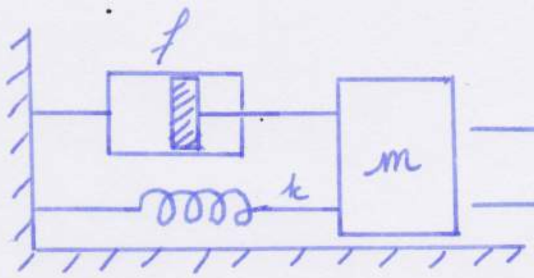
$$F = kx$$

$$F(t) = F_0 + k \int_0^t v\,dt$$

$$v = \frac{1}{k} \frac{dF}{dt}$$

$$dW = k x\,dx$$
$$= d\left(k \frac{x^2}{2}\right)$$

← potentiële energie

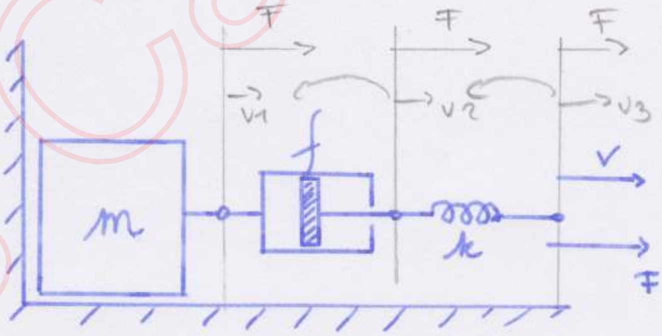


$$m a = \Sigma F \Leftrightarrow m\frac{dv}{dt} = F - fv - kx$$

$$m\frac{dv}{dt} + fv + kx = F$$

$$v = \frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx = F$$



$$v = v_1 + v_2 + v_3$$

$$\frac{dv_1}{dt} = \frac{F}{m}$$

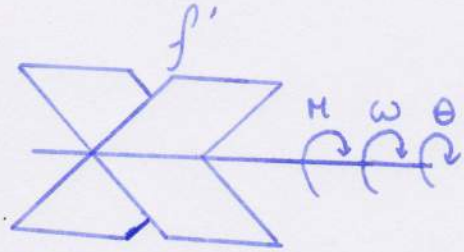
$$v_2 = \frac{F}{f}$$

$$v_3 = \frac{1}{k}\frac{dF}{dt}$$

$$\longrightarrow \frac{1}{k}\frac{d^2F}{dt^2} + \frac{1}{f}\frac{dF}{dt} + \frac{1}{m}F = \frac{dv}{dt}$$

Rotatie

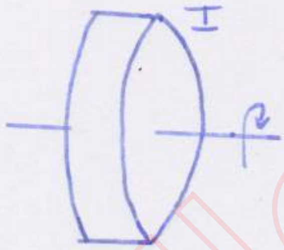
wrijvingsweerstand



$\omega = \frac{1}{f'} M \longleftarrow M = f'\omega$

$$dW = M\,d\theta$$
$$= f'\omega\,d\theta \longleftarrow \text{wrijvingsweerstand (warmte)}$$
$$= f'\omega^2 dt$$

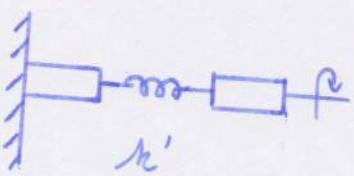
Traagheidsmoment



$\omega(t) = \omega_0 + \frac{1}{I}\int_0^t M\,dt \longleftarrow M = I\frac{d\omega}{dt}$

$dW = d\left(\frac{I\omega^2}{2}\right) \longleftarrow$ kin. en.

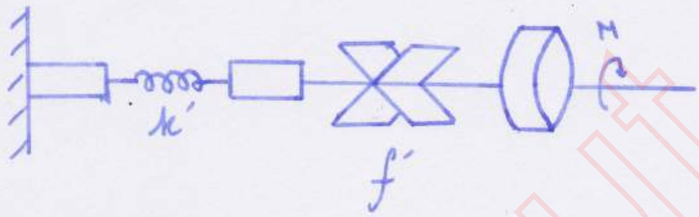
Torsie-veer



$\omega = \frac{1}{k'}\frac{dM}{dt} \longleftarrow M = k'\theta$

$M(t) = M_0 + k'\int_0^t \omega\,dt$

$dW = d\left(\frac{k'\theta^2}{2}\right) \longleftarrow$ pot. en.

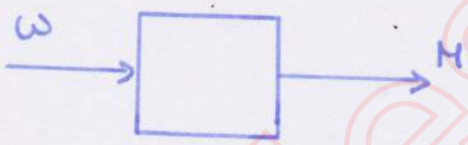


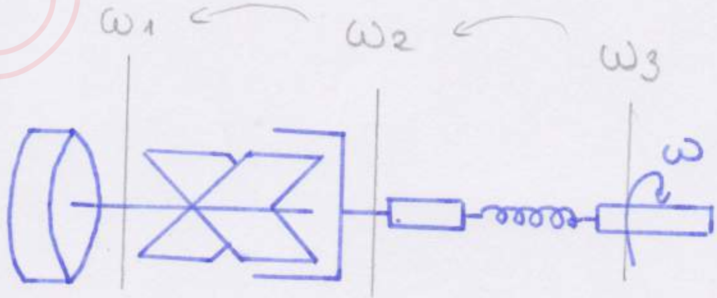
$$I \frac{d\omega}{dt} = \Sigma M = -k'\theta - f'\omega + M$$

$$\rightarrow \quad I \frac{d^2\omega}{dt^2} + f' \frac{d\omega}{dt} + k'\omega = \frac{dM}{dt}$$

$$\omega = \frac{d\theta}{dt}$$

$$I \frac{d^2\theta}{dt} + f' \frac{d\theta}{dt} + k'\theta = M$$





$$\omega = \omega_1 + \omega_2 + \omega_3$$

$$M = I \frac{d\omega_1}{dt} \qquad \omega_2 = \frac{1}{f'} M$$

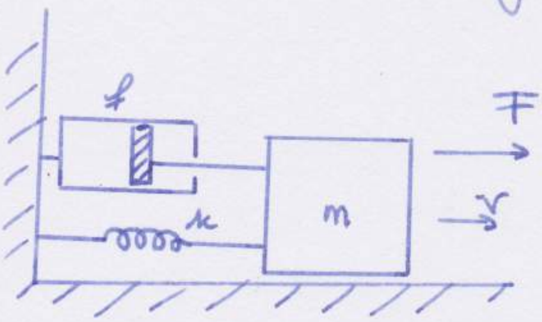
$$\omega_3 = \frac{1}{k'} \frac{dM}{dt}$$

$$\rightarrow \quad \frac{1}{k'} \frac{d^2M}{dt^2} + \frac{1}{f'} \frac{dM}{dt} + \frac{1}{I} M = \frac{d\omega}{dt}$$

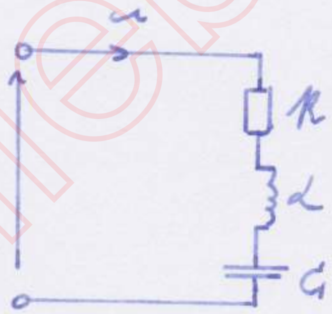
Analogie tussen systemen

$\longleftrightarrow$ + omrekeningsfactor

massa - veer - dempersysteem :



serie RLC - kring



$$m \frac{d^2 v}{dt^2} + f \frac{dv}{dt} + kv = \frac{dF}{dt}$$

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dv}{dt}$$

$\longleftrightarrow$

$$L = K_1 m$$

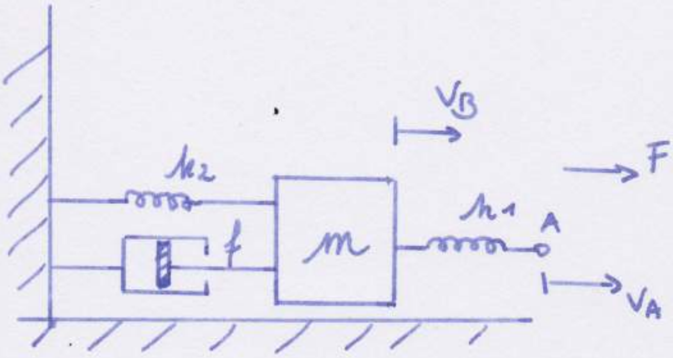
$$R = K_1 f$$

$$\frac{1}{C} = K_1 k$$

$$V(t) = K_2 F(t)$$

$$\longrightarrow \quad m \frac{d^2\left[\frac{K_1}{K_2} i\right]}{dt^2} + f \frac{d\left[\frac{K_1}{K_2} i\right]}{dt} + k\left[\frac{K_1}{K_2} i\right] = \frac{dF}{dt}$$

$$\longrightarrow \quad V(t) = \frac{K_1}{K_2} i(t)$$



veer 1: $v_A - v_B = \frac{1}{k_1}\frac{dF}{dt}$

massa: $m\frac{dv_B}{dt} + f v_B + k_2\int_0^t v_B\,dt = F$

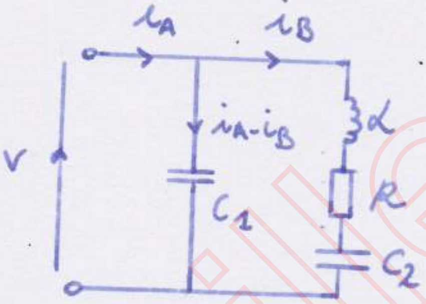
Analogie

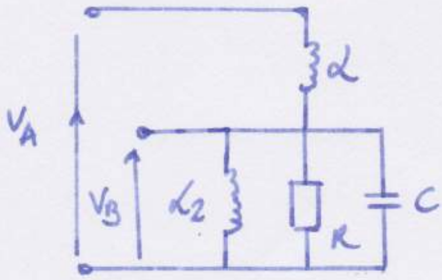
$$\mathcal{L}\frac{di_B}{dt} + R i_B + \frac{1}{C_2}\int_0^t i_B\,dt = v$$

$$C\frac{dv_B}{dt} + \frac{v_B}{R} + \frac{1}{\mathcal{L}_2}\int_0^t v_B\,dt = i$$

$$i_A - i_B = C_1\frac{dv}{dt}$$

$$v_A - v_B = \mathcal{L}_1\frac{di}{dt}$$





$$v_A(t) = \frac{K_1}{K_2}\, i_A(t)$$

$$v_B(t) = \frac{K_1}{K_2}\, i_B(t)$$

$$v_A(t) = \frac{K_1}{K_2}\, v_A(t)$$

snelheid

$$v_B(t) = \frac{K_1}{K_2}\, v_B(t)$$

snelheid

$$\frac{d}{d\tau} = \frac{1}{a}\frac{d}{dt}$$

$a < 1$: versneld

$a > 1$: vertraagd

Linearisatie

niet - lineaire verg $\xrightarrow{\text{Taylor}}$ L.T.O. systemen
als
bestuderen

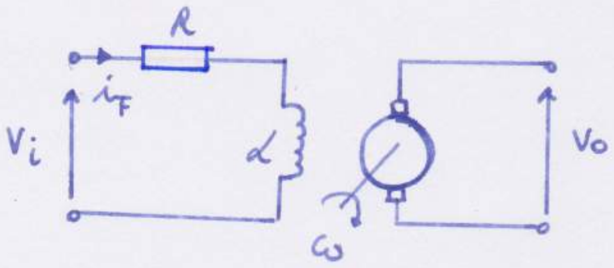
$f(x)$ berekening dicht bij x_0

$\rightarrow \quad f(x) = f(x_0) + \left.\frac{df}{dx}\right|_0 (x - x_0) + \dots$

als $(x - x_0)$ voldoende klein.

Eenvoudige elektromagnetische systemen

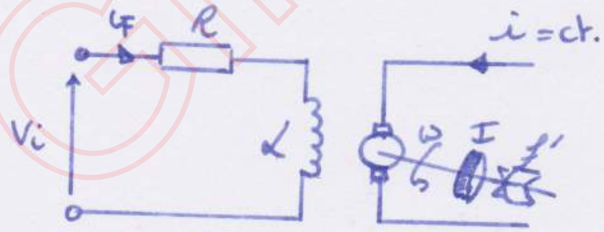
De Shunt-generator



$$\left.\begin{aligned} V_i &= R i_F + \mathcal{L}\frac{di_F}{dt} \\ V_o &= K i_F \end{aligned}\right\} \rightarrow \frac{\mathcal{L}}{K}\frac{dv_o}{dt} + \frac{R}{K} v_o = V_i$$

eerste orde systeem

Een belaste dc-motor



$$\mathcal{L}\frac{di_F}{dt} + R i_F = v_i$$

$$M = K i_F$$

$$M = f'\omega + I\frac{d\omega}{dt}$$

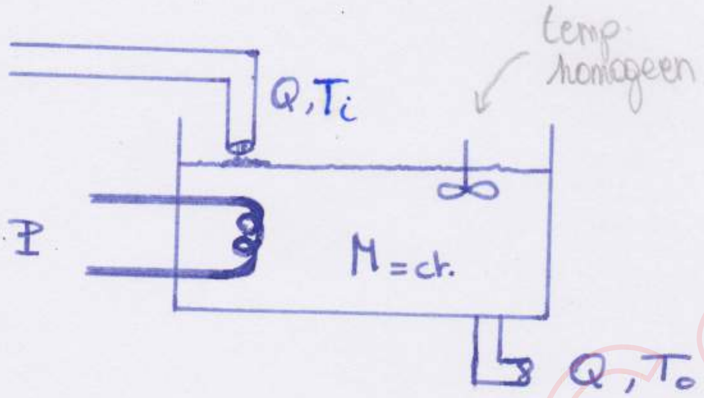
$$\downarrow$$

$$\frac{d^2\omega}{dt^2} + \left(\frac{f'}{I} + \frac{R}{\mathcal{L}}\right)\frac{d\omega}{dt} + \frac{Rf'}{\mathcal{L}I}\omega = \frac{K}{\mathcal{L}I} V_i$$

tweede orde systeem

Eenvoudige L.T.O. processen

Een verwarmingsketel



bij evenwicht:

$$\underbrace{P}_{\substack{\text{toegev.}\\ \text{warmte}}} = \underbrace{C_h}_{\substack{\text{soortelijke}\\ \text{warmte}}} . \underbrace{Q}_{\text{debiet}} . \underbrace{(T_o - T_i)}_{\substack{\text{temp.}\\ \text{verschil}}}$$

[J/s] [J/kg.K] . [kg/s] [K]

$$\rightarrow T_{oe} = T_{ie} + \frac{P}{C_h Q}$$

energiebalans in dt:

$$P\,dt = \underbrace{C_h Q (T_o - T_i) dt}_{\substack{\text{opwarmen}\\ \text{instromende}\\ \text{vloeistof}}} + \underbrace{C_h M\, dt}_{\substack{\text{opwarmen}\\ \text{vd ketel}\\ \text{in zijn geheel}}}$$

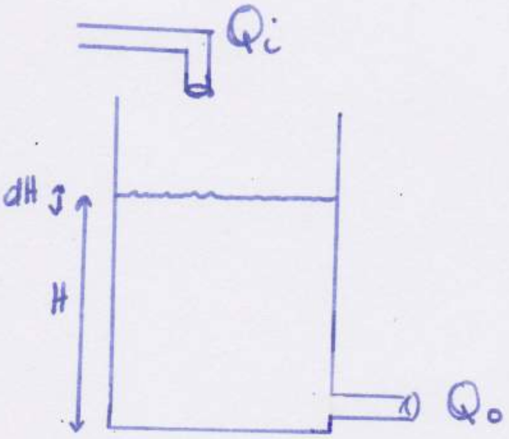
$$\rightarrow T_i = T_{ie} + t_i$$

$$T_o = T_{oe} + t_o$$

$$\Rightarrow P dt = C_h Q [(T_{oe} + t_o) - (T_{ie} + t_i)] dt + C_h M\, d(T_{oe} + t_o)$$

$$\Rightarrow \frac{M}{Q}\frac{dt_o}{dt} + t_o = t_i$$

Een vloeistoftank



$$Q_o = C\sqrt{H}$$

evenwicht : $Q_{ie} = Q_{oe} = C\sqrt{H_e}$

over kleine dt : $\underbrace{Q_i\,dt}_{\text{instromend debiet}} = \underbrace{Q_o\,dt}_{\text{uitstromend debiet}} + \underbrace{A\,dH}_{\text{aangroei hoeveelheid in tank}}$

variaties tov evenw. :
$$\left.\begin{aligned} Q_i &= Q_{ie} + q_i \\ Q_o &= Q_{oe} + q_o \\ H &= H_e + h \end{aligned}\right\}$$

$$\Rightarrow \quad A\frac{dH}{dt} + q_o = q_i \qquad (a)$$

verband h & q_o
nt-lineair : $Q_{oe} + q_o = C\sqrt{H_e + h}$

Taylor
$$Q_o(H) = Q_o(H_e) + \left.\frac{dQ_o}{dH}\right|_{H_e}(H - H_e)$$
$$Q_o(H) - \underbrace{Q_o(H_e)}_{Q_{oe}} = \left.\frac{dQ_o}{dH}\right|_{H_e}(H - H_e)$$

$$q_0 = \left.\frac{dQ_0}{dH}\right|_{He} \cdot h$$

$$q_0 = \left.\frac{d(C\sqrt{H})}{dH}\right|_{He} \cdot h$$

$$q_0 = \frac{C}{2\sqrt{He}} \cdot h = \frac{\overbrace{C\sqrt{He}}^{Q_{oe}}}{2He} \cdot h$$

$$q_0 = \frac{Q_{oe}}{2He} \cdot h \qquad \text{in (a)}$$

$$\Rightarrow \frac{2AHe}{Q_{oe}} \frac{dq_0}{dt} + q_0 = q_i$$

eerste orde systeem

Een gasreservoir

$$Q_o = C\sqrt{P}$$

$$Q_{ie} = Q_{oe} = C\sqrt{P_e}$$

$$\xrightarrow{Q_i} \boxed{P = ct.} \xrightarrow{Q_o\ [mol/s]}$$

algemene gaswet: $M_e = \dfrac{P_e V}{RT}$

massabalans: $Q_i\,dt = Q_o\,dt + \underbrace{dM}_{\text{hoeveelheid gas in reservoir}}$

evenw.:

$$\left.\begin{aligned} Q_i &= Q_{ie} + q_i \\ Q_o &= Q_{oe} + q_o \\ P &= P_e + p \\ M &= M_e + m \end{aligned}\right\} \rightarrow \frac{dm}{dt} + q_o = q_i$$

verband p en m:

$$\left.\begin{aligned} M_e + m &= \frac{(P_e + p)V}{RT} \\ M_e &= \frac{P_e V}{RT} \end{aligned}\right\} \rightarrow m = \frac{M_e}{P_e}\,p$$

$$\rightarrow \frac{M_e}{P_e}\frac{dp}{dt} + q_o = q_i$$

verband p en q_o: $Q_{oe} + q_o = C\sqrt{P_e + p}$

Taylor

$$p = \frac{2P_e}{Q_{oe}}\,q_o$$

$$\rightarrow \frac{2M_e}{Q_{oe}}\frac{dq_o}{dt} + q_o = q_i$$

eerste orde systeem

Besluit

eerste-orde systeem:

$$\tau \frac{dy(t)}{dt} + y(t) = K\,u(t)$$

tijdsconstante

statische versterkingsfactor

tweede-orde systeem:

dempingsgraad

$$\frac{1}{\omega_n^2}\frac{d^2y(t)}{dt^2} + \frac{2z}{\omega_n}\frac{dy(t)}{dt} + y(t) = K\,u(t)$$

eigen freq.

statische versterkingsfactor

Hoofdstuk 2

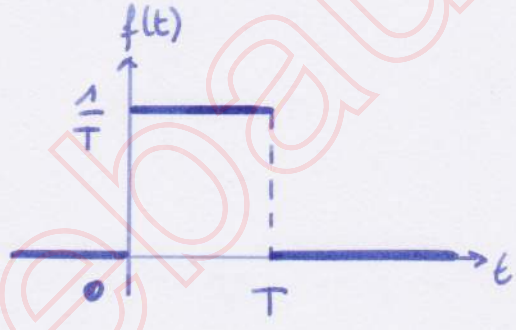
Studie van L.T.O systemen in het tijdsdomein

Samenvatting

Gilles Callebaut

Impulsresponsie

Impulsfunctie



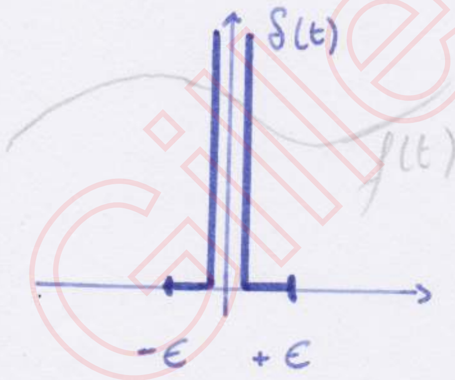
Diracfunctie: $\delta(t) = \lim_{T \to 0} f(t)$

$$\begin{cases} f(t) = 0 & t < 0 \\ f(t) = \frac{1}{T} & 0 \leq t \leq T \\ f(t) = 0 & t > T \end{cases}$$

↓

$$\int_{-\epsilon}^{+\epsilon} \delta(t)\, dt = 1$$

impulsfunctie met intensiteit 1



eigenschap: $\int_{-\epsilon}^{+\epsilon} f(t) . \delta(t)\, dt = f(0)$

Dirac op tijdstip τ: $\int_{\tau-\epsilon}^{\tau+\epsilon} \delta(t-\tau)\, dt = 1$

$$\int_{\tau-\epsilon}^{\tau+\epsilon} f(t)\, \delta(t-\tau)\, dt = f(\tau)$$

De Impulsresponsie

$$\underset{\text{éénheids-impuls}}{\xrightarrow{\delta(t)}} \boxed{} \underset{h(t)}{\xrightarrow{\text{impulsresponsie}}}$$

eerste-orde systeem: $\tau \dfrac{dy}{dt} + y(t) = K\, u(t)$

$y(t) = h(t)$, $u(t) = \delta(t)$

impulsresponsie:

$$h(t) = 0 \quad t < 0$$

$$\tau \frac{dh}{dt} + h = 0 \quad t > 0$$

$$\tau \Big[h(t) \Big]_{-\epsilon}^{+\epsilon} + \int_{-\epsilon}^{+\epsilon} h(t)\, dt = K \int_{-\epsilon}^{+\epsilon} \delta(t)\, dt \qquad \text{interval } [-\epsilon, +\epsilon]$$

$$\rightarrow \tau \Big[h(+\epsilon) - \underbrace{h(-\epsilon)}_{\substack{=0 \\ \text{want geen} \\ \text{responsie}}} \Big] + \underbrace{\int_{-\epsilon}^{+\epsilon} h(t)\, dt}_{\substack{=0 \\ \text{want } \lim_{\epsilon \to 0}}} = K \underbrace{\int_{-\epsilon}^{+\epsilon} \delta(t)\, dt}_{1}$$

$\lim_{\varepsilon \to 0}$

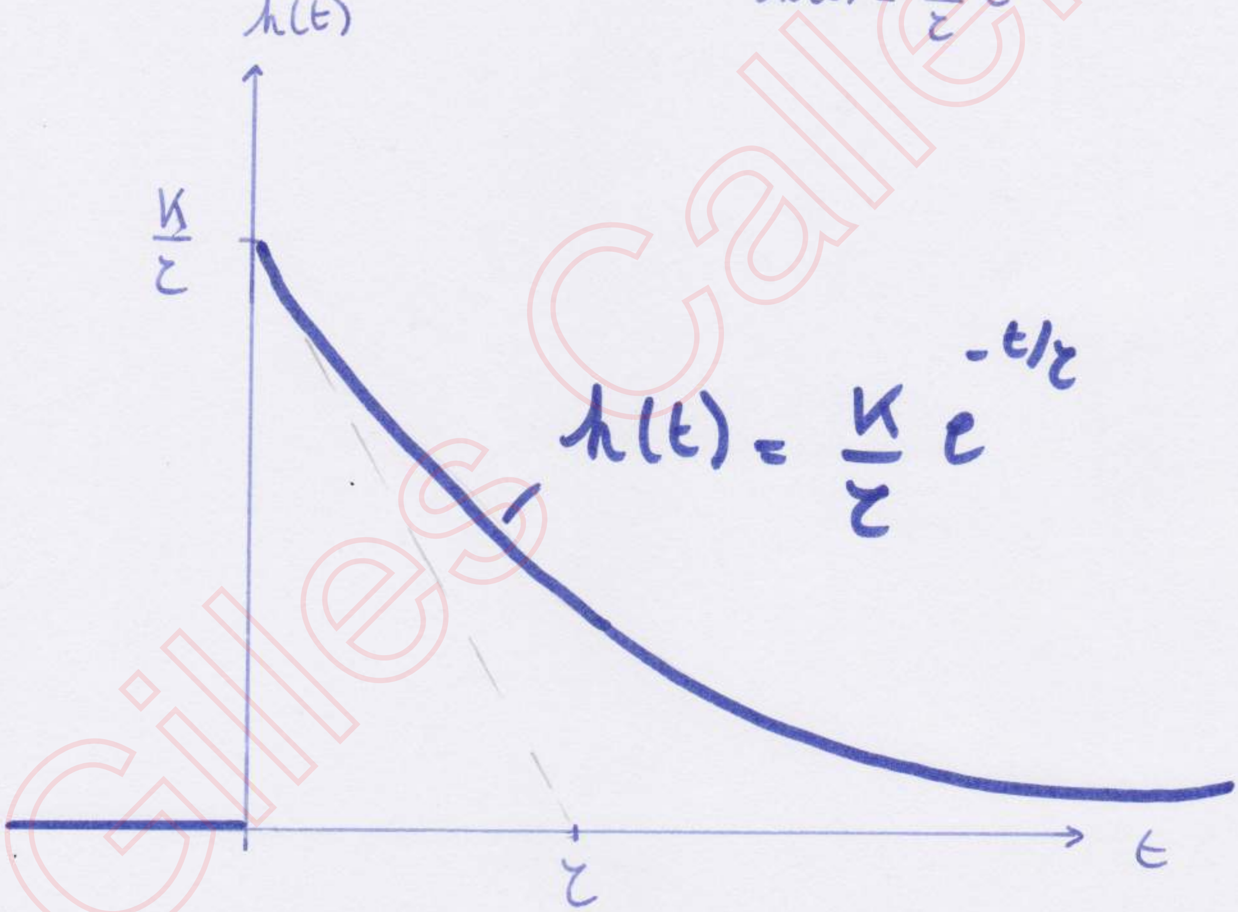
$$\rightarrow h(0^+) = \frac{K}{\tau} \qquad \text{(beginvoorwaarde)}$$

we weten: $\tau \dfrac{dh}{dt} + h = 0$ met beginVW: $h(0^+) = \dfrac{K}{\tau}$

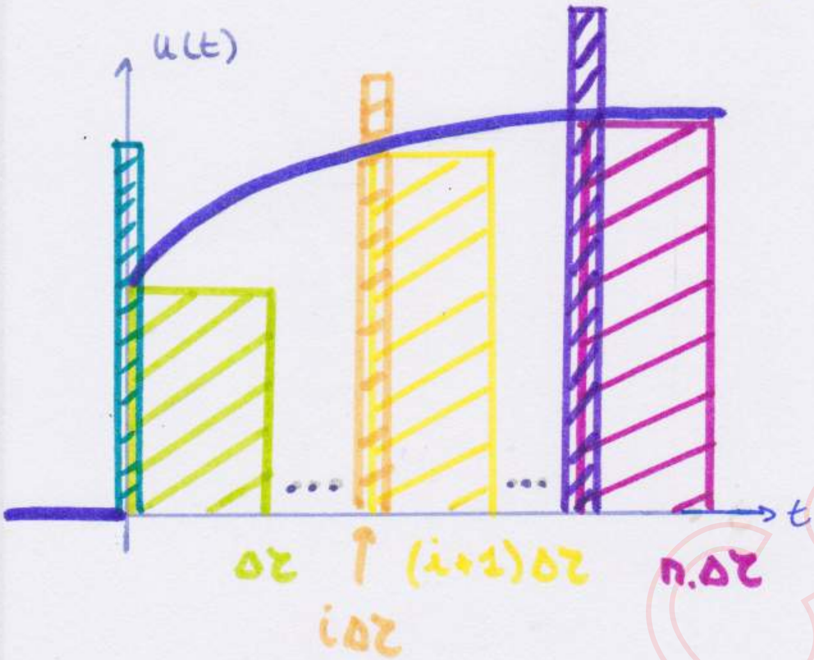
$\rightarrow \tau s + 1 = 0$

$\hookrightarrow s = -\dfrac{1}{\tau} \rightarrow h(t) = A e^{-t/\tau}$ met BVW

$h(t) = \dfrac{K}{\tau} e^{-t/\tau}$



De responsie op een willekeurige ingang



u(t) beschouwen als som van impulsfuncties.

↓

y(t) som v. alle responsies op de impulsfuncties.

Opp interval: $u(i\Delta\tau).\Delta\tau \xrightarrow{\text{begin}} u(i\Delta\tau)\,\Delta\tau\,\delta(t-i\Delta\tau)$

eigensch. dirac.

impulsresponsie: $u(i\Delta\tau)\Delta\tau\, h(t-i\Delta\tau)$

ingang: $\sum_{i=0}^{n-1} u(i\Delta\tau)\,\Delta\tau.\delta(t-\tau)$ uitgang: $\sum_{i=0}^{n-1} u(i\Delta\tau)\,\Delta\tau.\,h(t-i\Delta\tau)$

$t' = i\Delta\tau \qquad \Delta t' = \Delta\tau$

$$\sum_{t'=0}^{t-\Delta\tau} u(t')\,h(t-t')\,\Delta t'$$

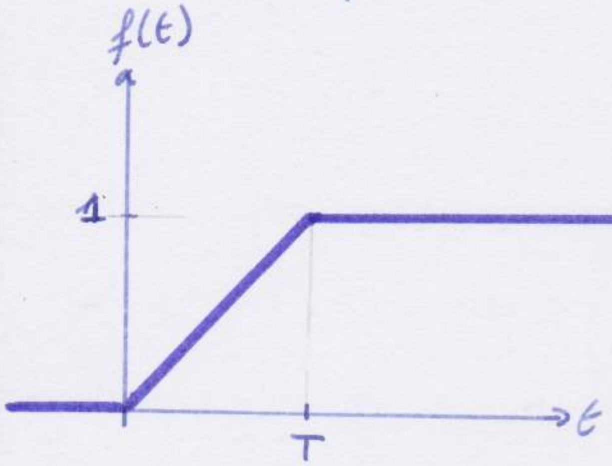
$\Delta t' \to dt'$

van de eerste soort

convolutie-integraal $\longrightarrow$ $$y(t) = \int_0^t u(t')\,h(t-t')\,dt'$$

Stapresponsie

De stapfunctie

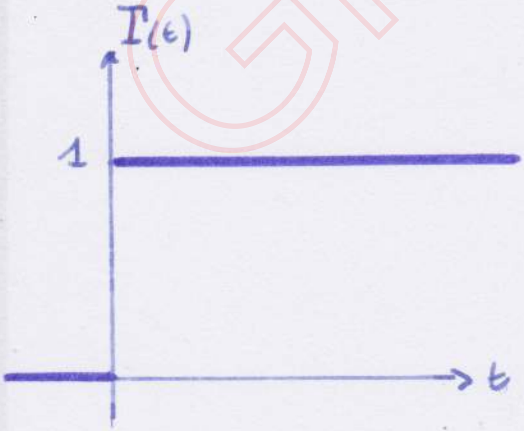


$$\begin{cases} f(t) = 0 & \text{voor } t < 0 \\ f(t) = \frac{t}{T} & \text{voor } 0 \leq t \leq T \\ f(t) = 1 & \text{voor } t > T \end{cases}$$

$$\rightarrow \begin{cases} \frac{df}{dt} = 0 & \text{voor } t < 0 \\ \frac{df}{dt} = \frac{1}{T} & \text{voor } 0 \leq t < T \\ \frac{df}{dt} = 0 & \text{voor } t > T \end{cases}$$

Heavyside:

$$\Gamma(t) = \lim_{T \to 0} f(t)$$



bij limietovergang: $\delta(t) = \frac{d\Gamma(t)}{dt}$

$f(t) \rightarrow \delta(t)$

$$\Gamma(t) = \int_{-\infty}^{t} \delta(t)\, dt$$

op tijdstip τ: $\Gamma(t-\tau) = 0$ voor $t < \tau$

$\Gamma(t-\tau) = 1$ voor $t \geq \tau$

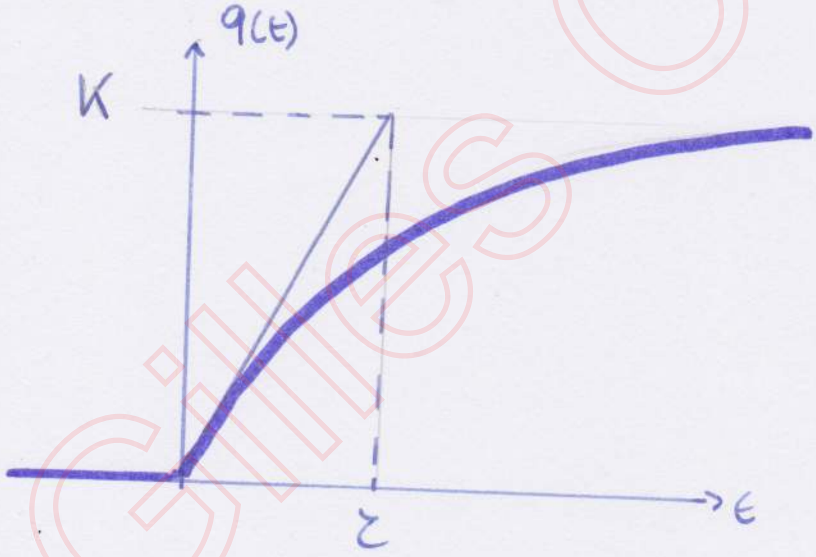
De stapresponsie

van de eerste soort

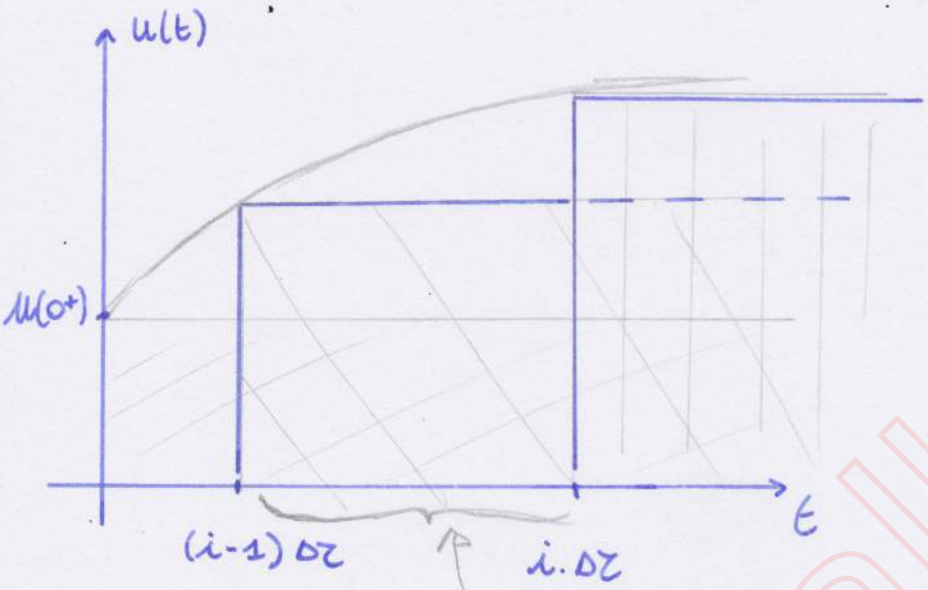
convolutie-integraal : $\Gamma(t) * h(t)$

$$\rightarrow \quad q(t) = \int_0^t 1 \cdot \frac{K}{\tau} e^{-t'/\tau} dt' = \left(\frac{K}{\tau} \cdot -\tau\right)\left[e^{-t'/\tau}\right]_0^t$$

$$q(t) = K\left(1 - e^{-t/\tau}\right)$$



de convolutie - integraal vd tweede soort



stapfunctie in het interval: $\left[u(i.\Delta\tau) - u((i-1)\Delta\tau)\right] . \Gamma(t - i\Delta\tau)$

↓

responsie: $\left[u(i\Delta\tau) - u((i-1)\Delta\tau)\right] . g(t - i\Delta\tau)$

ingang $u(t)$: $u(0^+)\,\Gamma(t) + \sum_{i=0}^{n} \left[u(i\Delta\tau) - u((i-1)\Delta\tau)\right] . \Gamma(t - i\Delta\tau)$

↓

responsie: $u(0^+)\,g(t) + \sum_{i=0}^{n} \left[u(i\Delta\tau) - u((i-1)\Delta\tau)\right] . g(t - i\Delta\tau)$

met $t' = i\Delta\tau$ is $\Delta t' = \Delta\tau$

en $\Delta\tau \to dt'$

$$\Rightarrow \quad y(t) = u(0^+) . g(t) + \int_0^t \frac{du(t)}{dt'}\, g(t - t')\, dt'$$

Harmonische responsie

Harmonische functies

pulsatie [rad/s]

$$F \cos(\omega t + \varphi) \xrightarrow{C.V.} \overline{F} = F e^{j\varphi}$$

amplitude — fase [rad] — modulus — argument

afleiden: $\overline{F}.j\omega$

integreren: $\overline{F}.\frac{1}{j\omega}$

$e^{j\varphi} = \cos\varphi + j\sin\varphi$

$$\tilde{f}(t) = F.\cos(\omega t + \varphi) + jF\sin(\omega t + \varphi)$$

$$= F e^{j(\omega t + \varphi)} = F e^{j\varphi} e^{j\omega t} = \overline{F}.e^{j\omega t}$$

een complexe harmonische functie $\tilde{f}(t)$ in complex vlak

De harmonische responsie

$$u(t) = U\cos(\omega t + \alpha) \longrightarrow y(t) = Y\cos(\omega t + \beta)$$

$$\overline{U} = U e^{j\alpha} \qquad \overline{Y} = Y e^{j\beta}$$

harmonische responsie = P.O. vd diff. verg:

$$B_n \frac{d^2 y}{dt^n} + \dots + B_0 y = A_m \frac{d^m u}{dt^m} + \dots + A_0 u$$

$$\downarrow \text{C.V.}$$

$$B_n (j\omega)^n \overline{Y} + \dots + B_0 \overline{Y} = A_m (j\omega)^m \overline{U} + \dots + A_0 \overline{U}$$

$$\Rightarrow \overline{Y} = H(j\omega)\,\overline{U} \quad \text{met} \quad H(j\omega) = \frac{A_m (j\omega)^m + \dots + A_0}{B_n (j\omega)^n + \dots + B_0}$$

complexe overdrachtsfunctie
of
complexe transferfunctie

$$\begin{cases} Y = |H(j\omega)|\, U \\ \arg \overline{Y} = \arg H(j\omega) + \arg \overline{U} \\ \beta = \arg H(j\omega) + \alpha \end{cases}$$

fase verschuiving van uitgang op ingang

Complexe overdrachtfunctie

1^{ste} orde:

$$\tau \frac{dy}{dt} + y = K.u$$

$$\longrightarrow \quad H(j\omega) = \frac{K}{1 + j\omega\tau}$$

2^{de} orde:

$$\frac{1}{\omega_n^2} \frac{d^2y}{dt^2} + \frac{2z}{\omega_n} \frac{dy}{dt} + y = K.u$$

$$\longrightarrow \quad H(j\omega) = \frac{K}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2z \frac{j\omega}{\omega_n} + 1}$$

De responsie op een willekeurig ingangssignaal

periodiek ingangssignaal → Fourierreeksontwikkeling

$$u(t) = u_0 + \sum_{h=1}^{\infty} U_h \cos(h\omega t + \alpha_h)$$

→ responsie : $\overline{Y}_h = H(j\omega h)\,\overline{U}_h$

$$Y_h e^{j\beta_h} \rightarrow \quad y(t) = y_0 + \sum_{h=1}^{\infty} Y_h \cos(h\omega t + \beta_h)$$

niet - periodiek ingangssignaal

$$\rightarrow \quad u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(j\omega)\, e^{j\omega t}\, d\omega$$

$$\text{met} \quad U(j\omega) = \int_{-\infty}^{+\infty} u(t)\, e^{-j\omega t}\, d\omega$$

→ responsie :

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(j\omega)\, U(j\omega)\, e^{j\omega t}\, d\omega$$

$$Y(j\omega) = H(j\omega)\, U(j\omega)$$

De grafische voorstelling vd complexe overdrachtfunctie

De Bode-diagramma's

Amplitude karakteristiek

$$|H(j\omega)|_{dB} = 20 \log |H(j\omega)|$$

1^{ste} orde: $$H(j\omega) = \frac{K}{1 + j\omega\tau}$$

modulus: $$|H(j\omega)|_{dB} = 20 \log K - 20 \log \sqrt{1 + \omega^2\tau^2}$$

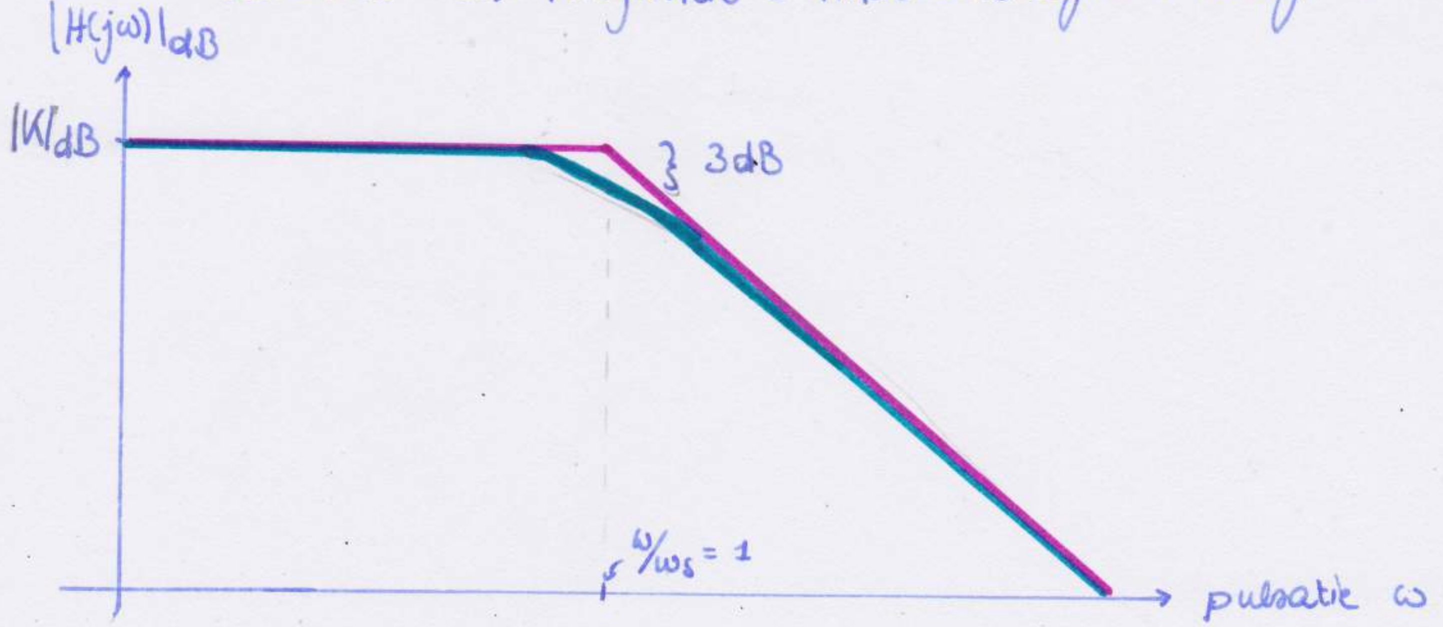
$\omega_s = \frac{1}{\tau}$ kritische freq.

$$= 20 \log K - 20 \log \sqrt{1 + \frac{\omega^2}{\omega_s^2}}$$

$\omega\tau \ll 1 \rightarrow |H(j\omega)|_{dB} = |K|_{dB}$

$\omega\tau = 1 \rightarrow |H(j\omega)|_{dB} = |K|_{dB} - 3$

$\omega\tau \gg 1 \rightarrow |H(j\omega)|_{dB} = |K|_{dB} - 20 \log \omega + 20 \log \omega_s$



fasekarakteristiek:

1^{ste} orde: $\frac{K}{1+j\omega\tau}$

argument: $\arg H(j\omega) = -\arg(1+j\omega\tau)$ met

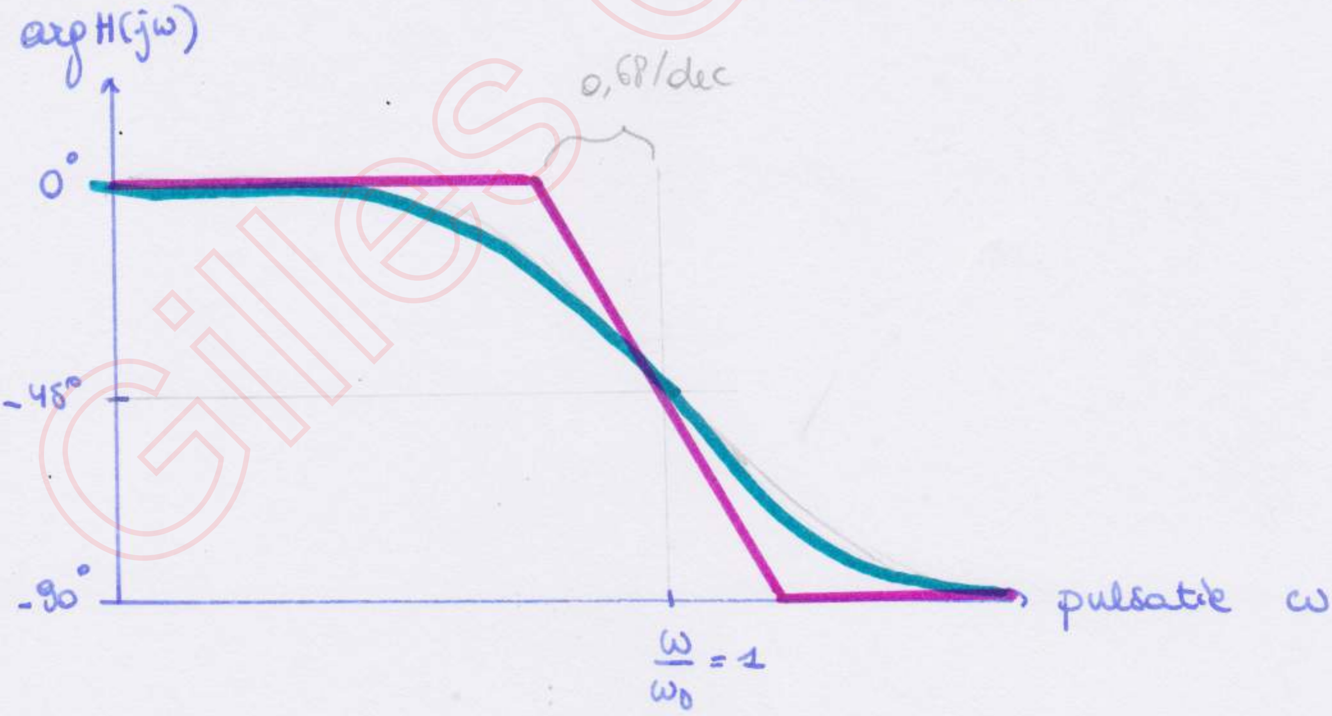
$K>0$: $\arg = 0$

$K<0$: $\arg = \pi$

$\omega\tau \ll 1$: $\arg H = 0$

$\omega\tau = 1$: $\arg H = -\arg(1+j) = -\frac{\pi}{4}$ $(-45°)$

$\omega\tau \gg 1$: $\arg H = -\arg(j\omega\tau) = -\frac{\pi}{2}$ $(-90°)$



voor een 2^{de} orde systeem:

$$H(j\omega) = \frac{K}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2z\frac{j\omega}{\omega_n} + 1} = \frac{K}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + j\left(2z\frac{\omega}{\omega_n}\right)}$$

$$|H(j\omega)|_{dB} = 20\log K - 20\log\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2z\frac{\omega}{\omega_n}\right)^2}$$

$\frac{\omega}{\omega_n} << 1 \rightarrow |H(j\omega)|dB = 0$ met $K = 1$

$\frac{\omega}{\omega_n} >> 1 \rightarrow |H(j\omega)|dB = -20\log\left(\left(\frac{\omega}{\omega_n}\right)^4\right) = -40\log\left(\frac{\omega}{\omega_n}\right)$

maximum bij: $\frac{d}{du}|H(j\omega)| = 0$

$$2(1-u^2)(-2u) + (2z)^2 2u = 0$$

$\hookrightarrow u\left[u^2 + 2z^2 - 1\right] = 0$

$u = 0$ $\qquad u = \sqrt{1-2z^2}$

met VW: $1 - 2z^2 > 0$

$z < \frac{1}{\sqrt{2}} \approx 0{,}707$

$\rightarrow \omega_R = \omega_n\sqrt{1-2z^2}$

$\rightarrow Q = \frac{1}{2z\sqrt{1-z^2}}$

maximale amplitude-verhouding

$Q = |H(j\omega)|_{\omega = \omega_R}$

fig 2.21
p51

$$\arg H(j\omega) = -\arg\left[(1-u^2) + j(2zu)\right]$$

$u = 1$: $\arg H = -\arg(j2z) = -\frac{\pi}{2}$

$u \ll 1$: $\arg H = -\arg(1) = 0$

$u \gg 1$: $\arg H = -\arg\left[(1-u^2) + j(2zu)\right] = -\pi$

bepaald door u^2

fig 2.22 en fig 2.23 p52

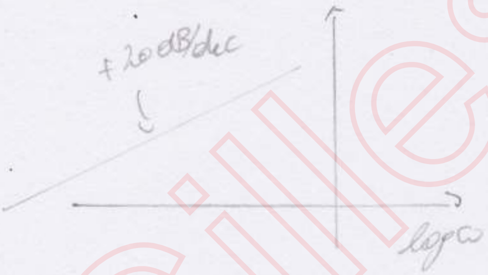
Algemeen

$$H(j\omega) = \frac{A_m (j\omega)^m + \ldots + A_0}{B_n (j\omega)^n + \ldots + B_0}$$

voor lage frequenties:

- $\left.\begin{array}{l} A_0 \neq 0 \\ B_0 \neq 0 \end{array}\right\} \rightarrow H(j\omega) = \dfrac{A_0}{B_0}$

- $\left.\begin{array}{l} B_0 \neq 0 \\ A_{k-1}, \ldots, A_0 = 0 \end{array}\right\} \rightarrow H(j\omega) = \dfrac{A_k}{B_0} (j\omega)^k$

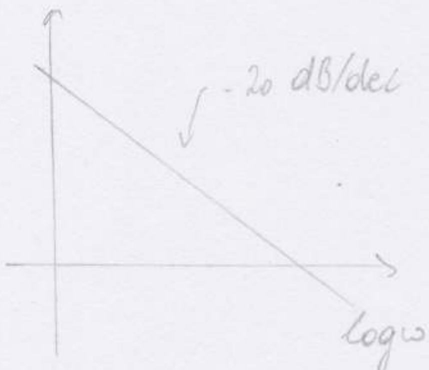


Modulus: $|H(j\omega)| \rightarrow 0$

ampl. kar. $k(+20)$ dB/dec

Argument: $\arg H(j\omega) = \arg \dfrac{A_k}{B_0} + k\dfrac{\pi}{2}$

- $\left.\begin{array}{l} A_0 \neq 0 \\ B_{k-1}, \ldots, B_k = 0 \end{array}\right\} \rightarrow H(j\omega) = \dfrac{A_0}{B_k} \dfrac{1}{(j\omega)^k}$



Modulus: $|H(j\omega)|$ vanuit ∞

ampl. kar. $k(-20)$ dB/dec

Argument: $\arg H(j\omega) = \arg \dfrac{A_0}{B_k} + k\left(-\dfrac{\pi}{2}\right)$

voor hoge frequenties:

$$H(j\omega) = \frac{A_m}{B_n} \frac{1}{(j\omega)^{n-m}}$$

- $n > m \rightarrow$ $|H(j\omega)| \mapsto 0$

  helling $(n-m)(-20 \text{ dB/dec})$

  $\arg H(j\omega) \mapsto (n-m)(-\frac{\pi}{2})$

- $n = m \rightarrow$ $H(j\omega) = \frac{A_m}{B_n}$

- $n < m \rightarrow$ fysische ~~werkelijkheid~~

  ↳ ingang steeds sneller

  ↳ uitgang niet meer kan volgen.

Opsplitsen transferfuncties:

$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$$

$$|H(j\omega)|_{dB} = |H_1(j\omega)|_{dB} + |H_2(j\omega)|_{dB}$$

$$\arg H = \arg H_1 + \arg H_2$$

$\sim$ zoeken vd vorm van

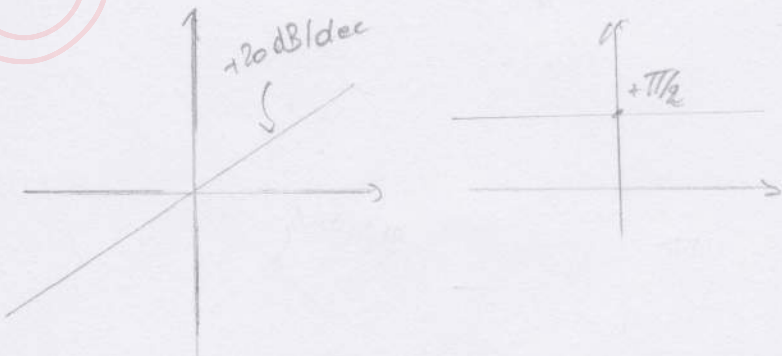
$1 + j\omega\tau$

of

$1 + \frac{2z}{\omega_n}(j\omega) + \frac{1}{\omega_n^2}(j\omega)^2$

factoren tegenkomen.

- $\frac{1}{j\omega}$ → integratiefactor



- $j\omega$ → differentiatiefactor

- $|K|_{dB}$ → verschuiving amplitude grafiek

$K < 0$ → faseverschuiving π

Het Nyquist-diagramma

↳ $H(j\omega)$ uitzetten $\overset{Im}{\llcorner}_{\to Re}$ met ω $0 \to \infty$

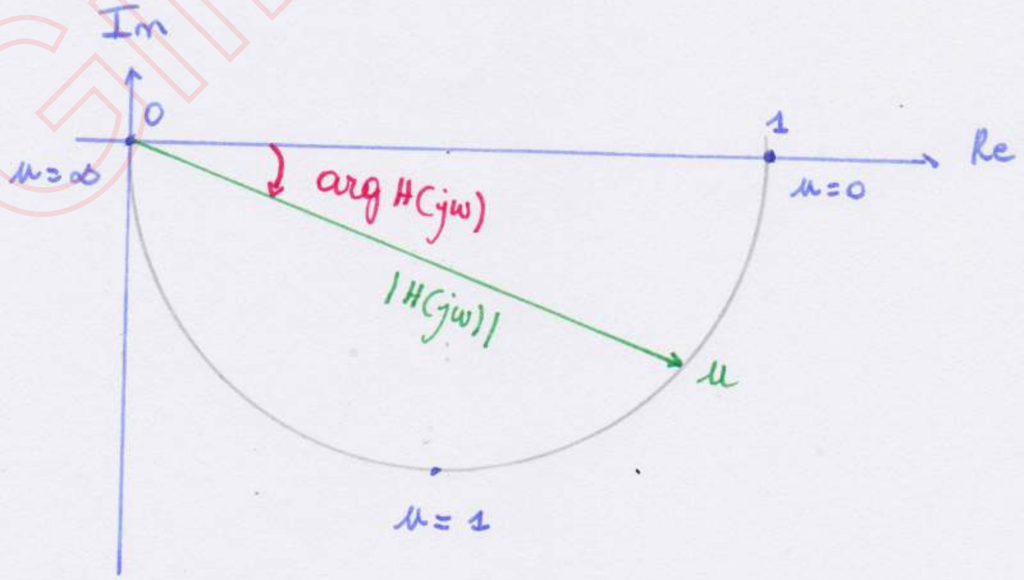
complex vlak

1^{ste} orde

$$H(ju) = \frac{1}{1+ju} \begin{cases} \nearrow Re = \dfrac{1}{1+u^2} \\ \searrow Im = \dfrac{-uj}{1+u^2} \end{cases}$$

$$Re^2 + Im^2 = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 \rightarrow \frac{1}{1+u^2} = Re^2 + Im^2$$

$$\Rightarrow \left(Re - \frac{1}{2}\right)^2 + Im^2 = \left(\frac{1}{2}\right)^2$$



2^{de} orde:

$$H(ju) = \frac{1}{1 + 2zju + (ju)^2} = \frac{1}{(1-u^2) + j2zu}$$

$$Re = \frac{1-u^2}{(1-u^2)^2 + 4z^2u^2} \qquad Im = \frac{-2zu}{(1-u^2)^2 + 4z^2u^2}$$

$u = 0 \rightarrow Re = 1 \quad Im = 0$

$u = 1 \rightarrow Re = 1 \quad Im = -\frac{1}{2z}$

$u = \infty \rightarrow Re = 0 \quad Im = 0$

- De Nyquist-krommen voor $z > 0{,}7$

  ↳ binnen $|H(ju)| = 1$

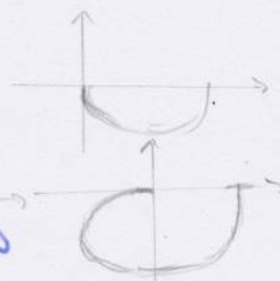
  voor $z < 0{,}7$

  ↳ resonantie

→ kleine freq. → loodrecht op Re-as.

→ hoge freq. → 1^{ste} orde: raken neg. Im as

→ 2^{de} orde: raken neg. Re as



Algemeen

$$H(j\omega) = \frac{A_m (j\omega)^m + \ldots + A_0}{B_n (j\omega)^n + \ldots + B_0}$$

hoge frequenties:

$\rightarrow H(j\omega) = \frac{A_m}{B_n} \frac{1}{(j\omega)^{n-m}}$ stel $n > m$

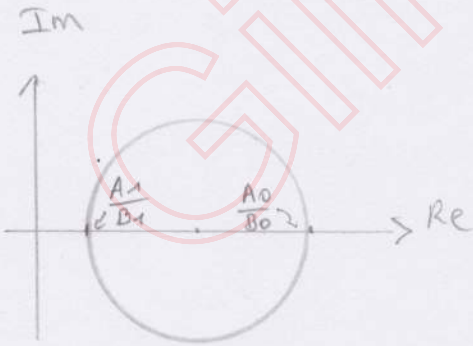
$|H(j\omega)| \rightarrow 0$

$\arg H(j\omega) = (n-m)(-\frac{\pi}{2})$

lage frequenties:

- $\left.\begin{array}{l} A_0 \neq 0 \\ B_0 \neq 0 \end{array}\right\} \Rightarrow H(j\omega) = \frac{A_0}{B_0} = K$

  $\hookrightarrow$ loodrecht vertrekken in K op reële as



zelfde verloop als

$$H(j\omega) = \frac{A_1 (j\omega) + A_0}{B_1 (j\omega) + B_0} \rightarrow M \left(\frac{1}{2}\left(\frac{A_0}{B_0} + \frac{A_1}{B_1}\right) ; 0 \right)$$

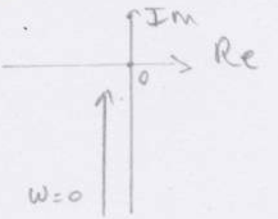
$$R = \frac{1}{2} \left| \frac{A_0}{B_0} + \frac{A_1}{B_1} \right|$$

- noemer: $j\omega$ voorop (int.)

$H(j\omega) = -j \frac{A_0}{B_0} \lim_{\omega \to 0} \frac{1}{\omega}$

$|H| \rightarrow \infty$

$\arg H \rightarrow -\frac{\pi}{2}$

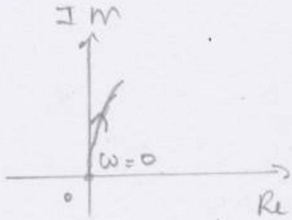


- teller: $j\omega$ voorop (diff.)

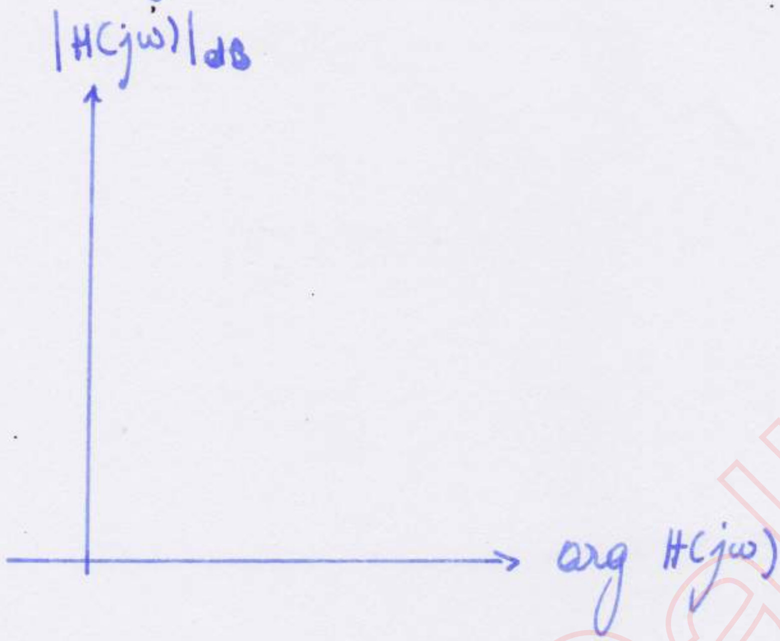
$H(j\omega) = j \frac{A_0}{B_0} \lim_{\omega \to 0} \omega$

$|H| \rightarrow 0$

$\arg H \rightarrow \frac{\pi}{2}$



Het diagramma van Black



met ω $0 \rightarrow \infty$

Hoofdstuk 3

De Laplace - transformatie

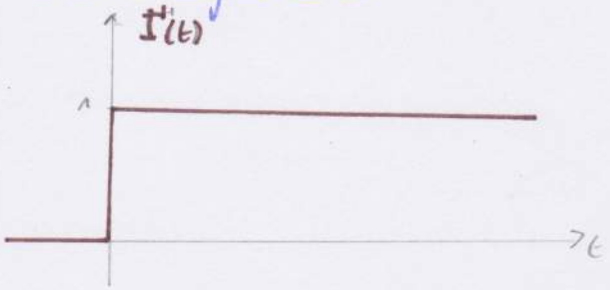
gilles callebaut

Definitie

$$\boxed{F(s) = \int_0^\infty f(t)\, e^{-st}\, dt} \qquad F(s) = \mathcal{L}\{f(t)\}$$

Laplace-getransformeerden van enkele eenvoudige functies

de Stapfunctie



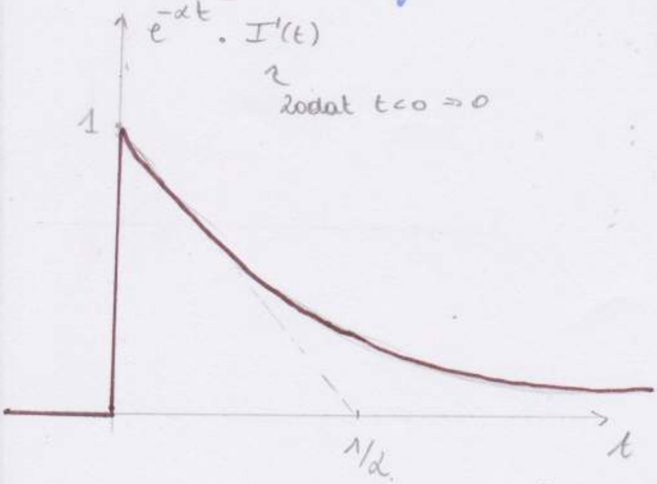
$$\mathcal{L}\{\Gamma(t)\} = \int_0^\infty 1 . e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty$$

$$\boxed{\mathcal{L}\{\Gamma(t)\} = \frac{1}{s}}$$

de impulsfunctie

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

de exponentiële functie



$$\mathcal{L}\{e^{-\alpha t}\} = \int_0^\infty e^{-\alpha t} . e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+\alpha)t}\, dt$$

$$= \left.\frac{e^{-(s+\alpha)t}}{-(s+\alpha)}\right|_0^\infty$$

$$\boxed{\mathcal{L}\{e^{-\alpha t}\} = \frac{1}{s+\alpha}}$$

Eigenschappen

lineariteit

Indien $F(s)_i \leftrightarrow f(t)_i$ $\nearrow$ $k\,F(s) \leftrightarrow k\,f(t)$
$\searrow$ $F_1(s) + F_2(s) \leftrightarrow f_1(t) + f_2(t)$

$$\Rightarrow \quad \mathcal{L}\left\{1 - e^{-\alpha t}\right\} = \frac{1}{s} - \frac{1}{s+\alpha} \quad \rightarrow \quad \boxed{\mathcal{L}\left\{1 - e^{-\alpha t}\right\} = \frac{\alpha}{s(s+\alpha)}}$$

$$\mathcal{L}\left\{1 - e^{-\frac{t}{\tau}}\right\} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}} \quad \rightarrow \quad \boxed{\mathcal{L}\left\{1 - e^{-t/\tau}\right\} = \frac{1}{s.(1+\tau.s)}}$$

$$\mathcal{L}\left\{\sin \omega t\right\} = \mathcal{L}\left\{\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right\} \rightarrow \quad \boxed{\mathcal{L}\left\{\sin \omega t\right\} = \frac{\omega}{s^2 + \omega^2}}$$

$$\mathcal{L}\left\{\cos \omega t\right\} = \mathcal{L}\left\{\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right\} \rightarrow \quad \boxed{\mathcal{L}\left\{\cos \omega t\right\} = \frac{s}{s^2 + \omega^2}}$$

$$\mathcal{L}\left\{\sin \omega t\right\} = \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right] = \frac{\omega}{s^2 + \omega^2}$$

de Afgeleide

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\,F(s) - f(0^+)$$

$$\Downarrow$$

$$\frac{d^2 f(t)}{dt^2} \leftrightarrow s^2F(s) - sf(0^+) - \frac{df(0^+)}{dt}$$

want

$$\mathcal{L}\{f''\} = \mathcal{L}\{f'\} - f'(0^+)$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt}\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st} \cdot df(t)$$

$$= f(t)\, e^{-st}\Big|_0^\infty - \int_0^\infty f(t) \cdot \frac{d(e^{-st})}{dt}$$

$$= -f(0) + s \underbrace{\int_0^\infty f(t)\, e^{-st}\, dt}_{F(s)}$$

de Integraal

$$\mathcal{L}\left\{\int_0^t f(t)\, dt\right\} = \frac{\mathcal{L}\{f(t)\}}{s}$$

$$\mathcal{L}\left\{\int_0^t f(t)\, dt\right\} = \int_0^\infty \int_0^t f(t)\, dt \cdot e^{-st}\, dt$$

$$= \int_0^\infty \int_0^t f(t)\, dt \cdot \frac{d(e^{-st})}{-s}$$

$\int$ P.I.

$$= \underbrace{\int_0^t f(t)\, dt \cdot \frac{e^{-st}}{-s}\Big|_\infty^0}_{\frac{1}{-s}[0-0]} + \underbrace{\int_0^\infty \frac{e^{-st}}{s} \cdot f(t)\, dt}_{\frac{1}{s}\mathcal{L}\{f(t)\}}$$

via de integraal eigenschap:

$$t = \int_0^t \Gamma(t)\,dt \Rightarrow \mathcal{L}\left\{t.\Gamma(t)\right\} = \frac{1}{s}\left(\frac{1}{s}\right)$$

$$t = \frac{1}{2}\frac{d(t^2)}{dt} \Rightarrow \mathcal{L}\left\{2\int_0^t\int_0^t \Gamma(t)\,dt\right\} = \mathcal{L}\left\{2\int_0^t t.\Gamma(t)\,dt\right\}$$

$$= 2\,\frac{1}{s}\,\frac{1}{s^2}$$

$\Rightarrow$ algemeen:

$$\boxed{\mathcal{L}\left\{t^n\,\Gamma(t)\right\} = \frac{n!}{s^n+1}}$$

beginwaardestelling

$$F(s) \sqsubset f(t) \rightarrow f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)\ ^{*}$$

eindwaardestelling

$$F(s) \sqsubset f(t) \rightarrow f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)\ ^{**}$$

$$^{*}\ \lim_{s\to\infty}\left[\int_0^\infty \frac{df}{dt}\,\underbrace{e^{-st}}_{=0}\,dt\right] \Rightarrow \lim_{s\to\infty}\left(sF(s) - f(0^+)\right) = 0$$

$$^{**}\ \lim_{s\to 0}\left[\int_0^\infty \frac{df}{dt}\cdot\underbrace{e^{-st}}_{=1}\,dt\right] \Rightarrow \lim_{t\to\infty}\underbrace{\int_0^\infty df(t)}_{f(\infty)-f(0^+)=} = \lim_{s\to 0}\left(sF(s) - f(0^+)\right)$$

Translatie in het s-domein

$$F(s) \ [\ f(t) \quad \rightarrow \quad F(s+\alpha) \ [\ e^{-\alpha t} f(t)$$

$$\mathcal{L}\left\{e^{-\alpha t}.f(t)\right\} = \int_0^\infty e^{-\alpha t} f(t) . e^{-st} \, dt$$

$$= \int_0^\infty f(t) \, e^{-\overbrace{(s+\alpha)}^{s'} t} dt$$

$$= F(s')$$

$$\boxed{\mathcal{L}\left\{e^{-\alpha t} f(t)\right\} = F(s+\alpha)}$$

Vermenigvuldigen met t

$$\mathcal{L}\left\{t.f(t)\right\} = \int_0^\infty t.f(t).e^{-st} \, dt$$

$$= \int_0^\infty f(t).\left(-\frac{d(e^{-st})}{ds}\right) dt$$

$$= -\frac{d}{ds}\left[\int_0^\infty f(t).e^{-st} \, dt\right]$$

$$\boxed{\mathcal{L}\left\{t.f(t)\right\} = -\frac{d}{ds} F(s)}$$

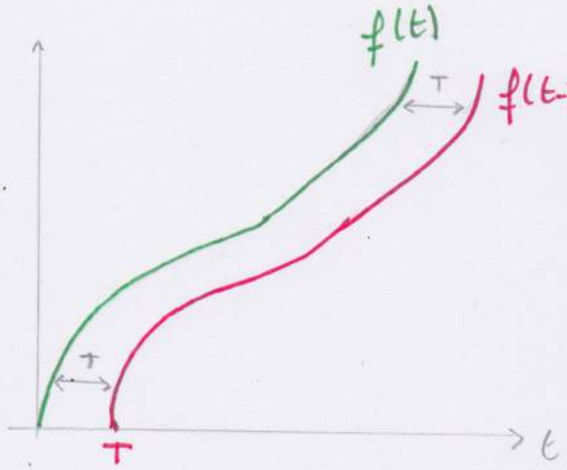
gelijkvormigheidswet

$$\mathcal{L}\left\{f\left(\tfrac{t}{k}\right)\right\} = \int_0^\infty f\left(\frac{t}{k}\right) . e^{-\overbrace{k.s}^{s'} . \overbrace{\frac{t}{k}}^{t'}} . dt$$

$$= k . \int_0^\infty f(t') . e^{-s't'} dt' = k \, F(s')$$

$$\boxed{\mathcal{L}\left\{f\left(\frac{t}{k}\right)\right\} = k . F(k.s)}$$

Verschuivingswet (in tijdsdomein)



$F(s)\,[f(t)\cdot I'(t)$

$$\boxed{\mathcal{L}\left\{f(t-T)\right\} = e^{-sT}\,F(s)}$$

$$\mathcal{L}\left\{f(\underbrace{t-T}_{T'})\,I'(t-T)\right\} = \int_0^{\infty} f(t')\,I'(t')\,e^{-s(t'+T)}\,dt'$$

$t' = t-T$
$dt' = dt$
$t'=0 \Rightarrow t=T$
$t'=\infty \Rightarrow t=\infty$

$$= e^{-sT}\cdot\int_0^{\infty} f(t')\,e^{-st'}\,dt'$$

Convolutiestelling

$y(t) = f(t) * g(t)$ ← convolutie

convolutie-stelling

$$\Rightarrow y(t) = \int_0^t f(\tau)\cdot g(t-\tau)\,d\tau = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$$G(s)\cdot F(s) = G(s)\cdot\int_0^{\infty} f(\tau)\,e^{-s\tau}\,d\tau$$

$$= \int_0^{\infty} f(\tau)\,\underbrace{G(s)\,e^{-s\tau}}_{[\int_0^{\infty} g(t-\tau)\,e^{-st}\,dt}\,d\tau$$ ← zie verschuivingswet

$$= \int_0^{\infty}\underbrace{\int_0^{\infty} f(\tau)\,g(t-\tau)\,d\tau}_{y(t)}\cdot e^{-st}\,dt$$

$$\boxed{\mathcal{L}\left\{g(t) * f(t)\right\} = G(s)\cdot F(s)}$$

De Inverse Laplace-transformatie

oef. !

vd vorm $F(s) = \frac{T(s)}{N(s)}$

→ splitsen in partieelbreuken:

voor meervoudige wortel

vd vorm $\frac{k_1}{s-p} + \frac{k_2}{(s-p)^2} + \frac{k_3}{(s-p)^3} + \cdots$

algemeen: $\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \frac{k_3}{s-p_3} + \cdots$

polen

vb. zie p71.

andere manier:

enkel voor enkelvoudige polen!

→ bepalen vd residu's $k = F(s)(s-p)\big|_{s=p}$

opmerking:

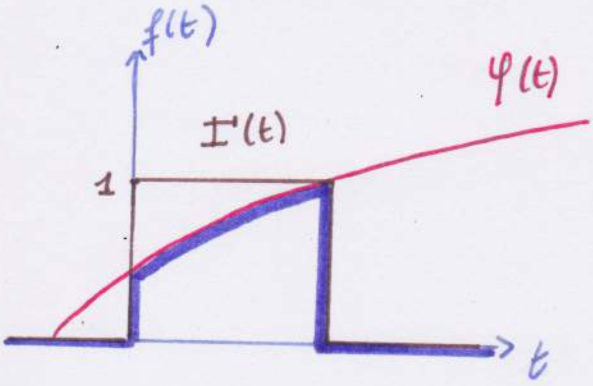
$$\frac{k_1 \cdot s + k_2}{(\ldots s^2 + \ldots s + \ldots)}$$

eerste graad

2de graad

Toepassingen

toepassingen op de verschuivingswet



$f \neq 0 \quad (0,T)$

de rest $f(t) = 0$

$$f(t) = \varphi(t)\left[I'(t) - I'(t-T)\right]$$

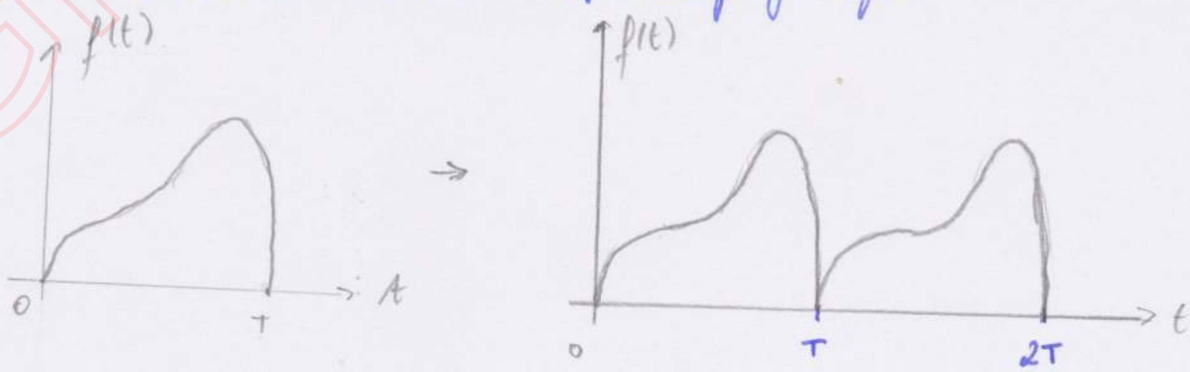
$$f(t) = \varphi(t)\, I'(t) - \varphi(t)\, I'(t-T)$$

als we $\varphi(t)$ schrijven als $\varphi[(t-T)+T]$

$$\Rightarrow \boxed{f(t) = \varphi(t)\, I'(t) - \varphi(t-T)\, I'(t-T) - \varphi(T)\, I'(t-T)}$$

- De Laplace-getransformeerde van periodieke functies

$f(t)$ die nul is buiten $[0,T]$ + opeenvolging zelfde verloop



$$f(t) = f(t) \cdot I'(t) + \ldots + f(t-iT)\, I'(t-iT)$$

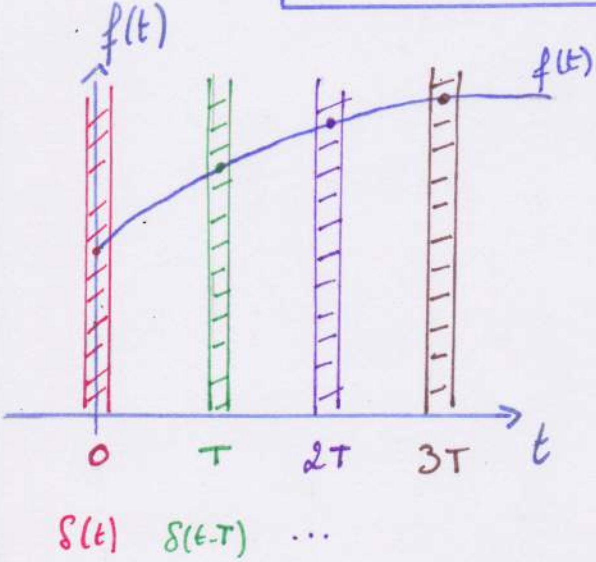
$$\boxed{f_T(t) = \sum_{i=0}^{\infty} f(t-iT)\, I'(t-iT)}$$

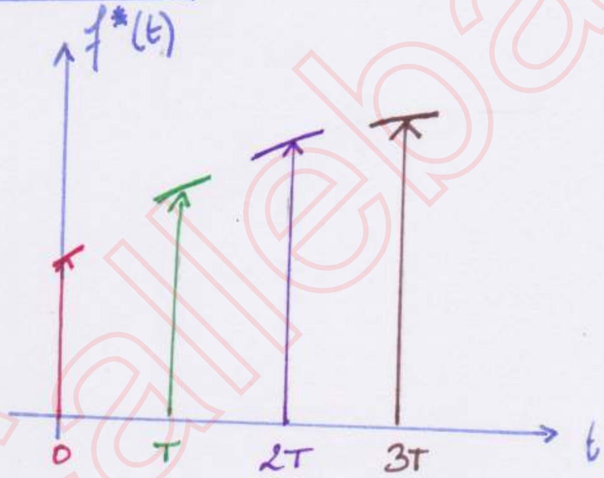
$$\boxed{F_T(s) = F(s) \sum_{i=0}^{\infty} e^{-iTs}}$$

$$\sum_{i=0}^{\infty} e^{-iTs} = \frac{1}{1-e^{-iTs}}$$

• De Laplace-getransformeerde van 'ideaal bemonsterde' functies

$$\boxed{f^*(t) = \sum_{i=0}^{\infty} f(iT)\,\delta(t - iT)}$$





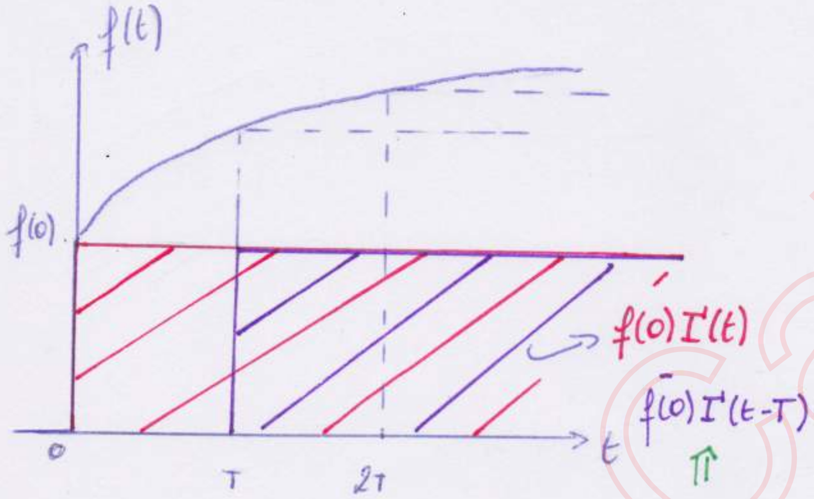
$$F^*(s) = \sum_{i=0}^{\infty} f(iT)\, e^{-iTs}$$

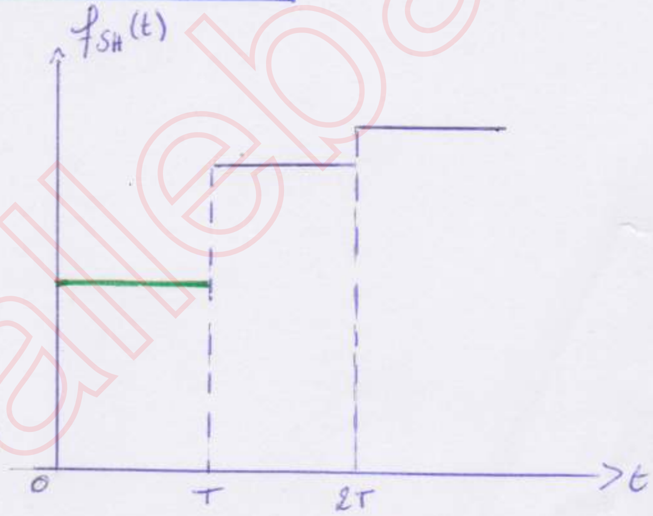
$$\begin{cases} z = e^{Ts} \\ e^{-iTs} = (e^{Ts})^{-i} \end{cases}$$

$$\boxed{F(z) = \sum_{i=0}^{\infty} f(iT)\, z^{-i}}$$

- De Laplace-getransformeerde van 'sample and hold'-functies

$$\boxed{f_{SH}(t) = \sum_{i=0}^{\infty} f(iT)\left[\Gamma(t-iT) - \Gamma(t-(i+1)T)\right]}$$





$$F_{SH}(s) = \sum_{i=0}^{\infty} f(iT)\left[\frac{e^{-iTs}}{s} - \frac{e^{-(i+1)Ts}}{s}\right]$$

$$F_{SH}(s) = \frac{1-e^{-Ts}}{s} \sum_{i=0}^{\infty} f(iT)\, e^{-iTs}$$

$$\boxed{F_{SH}(s) = \frac{1-e^{-Ts}}{s} F^{*}(s)}$$

Oplossen van lineaire differentiaalverg.

2^{de} orde:

$$B_2 \frac{d^2y(t)}{dt^2} + B_1 \frac{dy(t)}{dt} + B_0 \, y(t) = A_0 \, u(t)$$

$$\hookrightarrow B_2 \left[s^2 Y(s) - s\,y(0^+) - y'(0^+) \right] + B_1 \left[s\,Y(s) - y(0^+) \right] + B_0 \, Y(s) = A_0 \, U(s)$$

$$\rightarrow \quad Y(s) \left[s^2 B_2 + s B_1 + B_0 \right] = A_0 U(s) + (B_2 s + B_1) y(0^+) + B_2 \, y'(0^+)$$

$$Y(s) = \frac{A_0 \, U(s)}{s^2 B_2 + s B_1 + B_0} + \frac{B_2 s + B_1}{s^2 B_2 + s B_1 + B_0} y(0^+) + \frac{B_2}{s^2 B_2 + s B_1 + B_0} y'(0^+)$$

Hoofdstuk 4

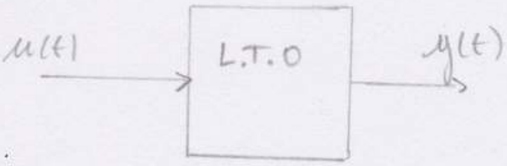
Studie van L.T.O. systemen

met hun

transferfunctie

Gilles Callebaut

De transferfunctie van een L.T.O. systeem



impulsresponsie

$$y(t) = h(t) * u(t)$$

$$\downarrow$$

$$Y(s) = H(s) \cdot U(s)$$

1^{ste} orde:

$$\boxed{H(s) = \frac{K}{1+s\tau}}$$

$$\tau \frac{dy(t)}{dt} + y(t) = K.u(t)$$

$$\hookrightarrow \tau.(sY(s)) + Y(s) = K.U(s)$$

mits $y(0^+) = 0$

2^{de} orde:

$$\boxed{H(s) = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1}}$$

$$\frac{1}{\omega_n^2}\frac{d^2y(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy(t)}{dt} + y(t) = K.u(t)$$

$$\hookrightarrow \frac{1}{\omega_n^2}\left(s^2Y(s)\right) + \frac{2\zeta}{\omega_n}\left(sY(s)\right) + Y(s) = K.U(s)$$

mits $y(0^+) = y'(0^+) = 0$

Het polen-nullen-diagramma van de transferfunctie (PND)

Algemeen

$$H(s) = \frac{A_m s^m + \ldots + A_1 s + A_0}{B_n s^n + \ldots + B_1 s + B_0}$$

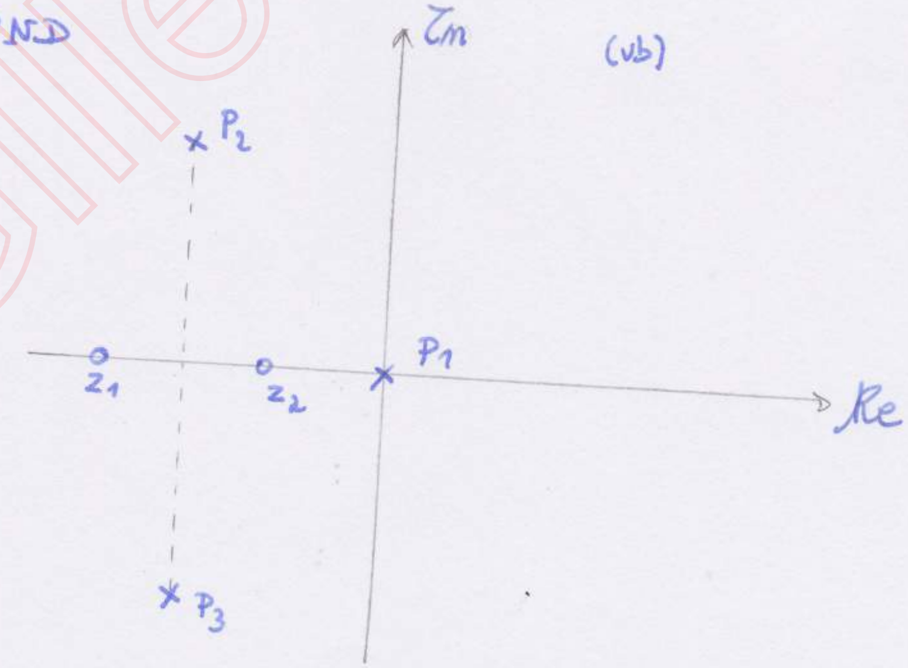
$$= \frac{A_m}{B_n} \cdot \frac{s^m + \cdots + a_0}{s^n + \cdots + b_0} = K \cdot \frac{T(s)}{N(s)}$$

$$= K \cdot \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

→ m nullen

→ n polen

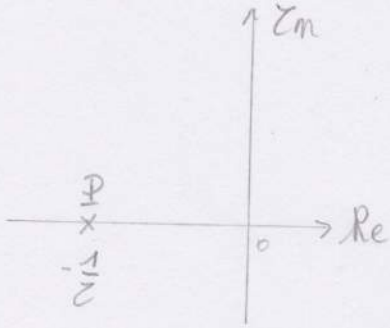
PND



Het PND van een eerste-orde systeem

$$H(s) = \frac{K}{1+s\tau} \rightarrow \frac{K}{\tau}\cdot\frac{1}{s+\frac{1}{\tau}}$$

$p_1 = -\frac{1}{\tau}$

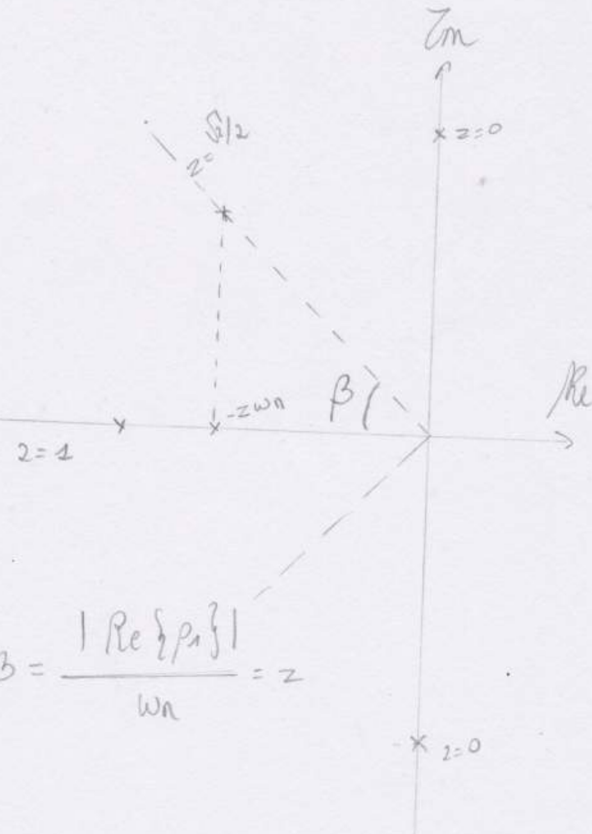


Het PND van een tweede-orde systeem

$$H(s) = \frac{K}{\left(\frac{s}{\omega_n}\right)^2 + \left(\frac{s}{\omega_n}\right)2z + 1} \rightarrow \frac{K\omega_n^2}{s^2 + 2z\omega_n s + \omega_n^2} \rightarrow \frac{K\omega_n^2}{(s+z\omega_n)^2 + \omega_n^2(1-z^2)}$$

wortels: $s^2 + 2z\omega_n s + \omega_n^2 = 0$

- $z > 1 \rightarrow p_1 = -\omega_n z + \omega_n\sqrt{z^2-1}$
  $\rightarrow p_2 = -\omega_n z - \omega_n\sqrt{z^2-1}$
- $z = 1 \rightarrow p_1 = p_2 = -\omega_n$
- $z < 1 \rightarrow p_{1,2} = -\omega_n z \pm j\omega_n\sqrt{1-z^2}$
- $z = 0 \rightarrow p_{1,2} = \pm j\omega_n$



$$\cos\beta = \frac{|Re\{p_1\}|}{\omega_n} = z$$

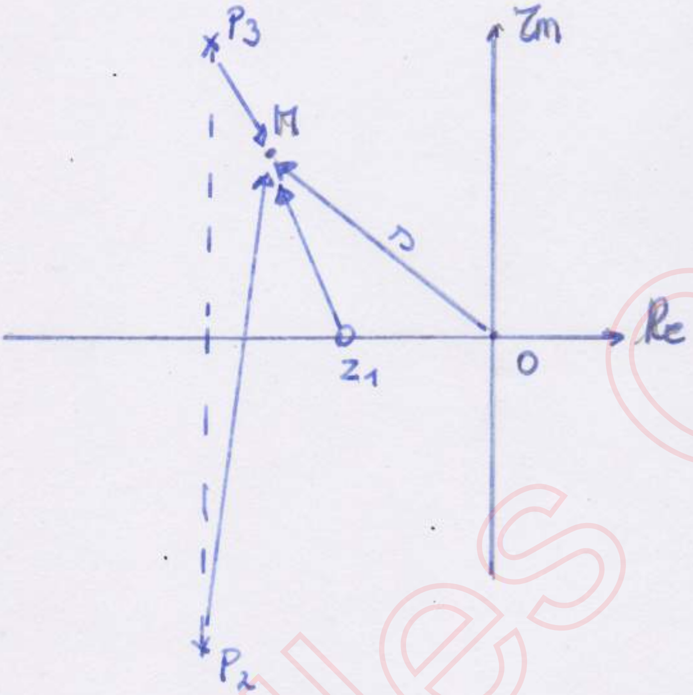
Grafische interpretatie van de transferfunctie i.h. PND

vectoren vanuit de nullen naar M

$$H(s) = \frac{(s-z_1)\dots(s-z_m)}{(s-p_1)\dots(s-p_n)} \cdot K$$

M

vectoren vanuit de polen naar M



modulus → lengte
argument → hoek

$$|H(s)| = \frac{|\overrightarrow{Z_1M}| \dots |\overrightarrow{Z_mM}|}{|\overrightarrow{P_1M}| \dots |\overrightarrow{P_nM}|} K$$

$$\arg H = (\arg \overrightarrow{z_1M} + \dots + \arg \overrightarrow{Z_mM}) - (\arg \overrightarrow{P_1M} + \dots + \arg \overrightarrow{P_nM})$$

$k > 0$

De impulsresponsie van een L.T.O. systeem

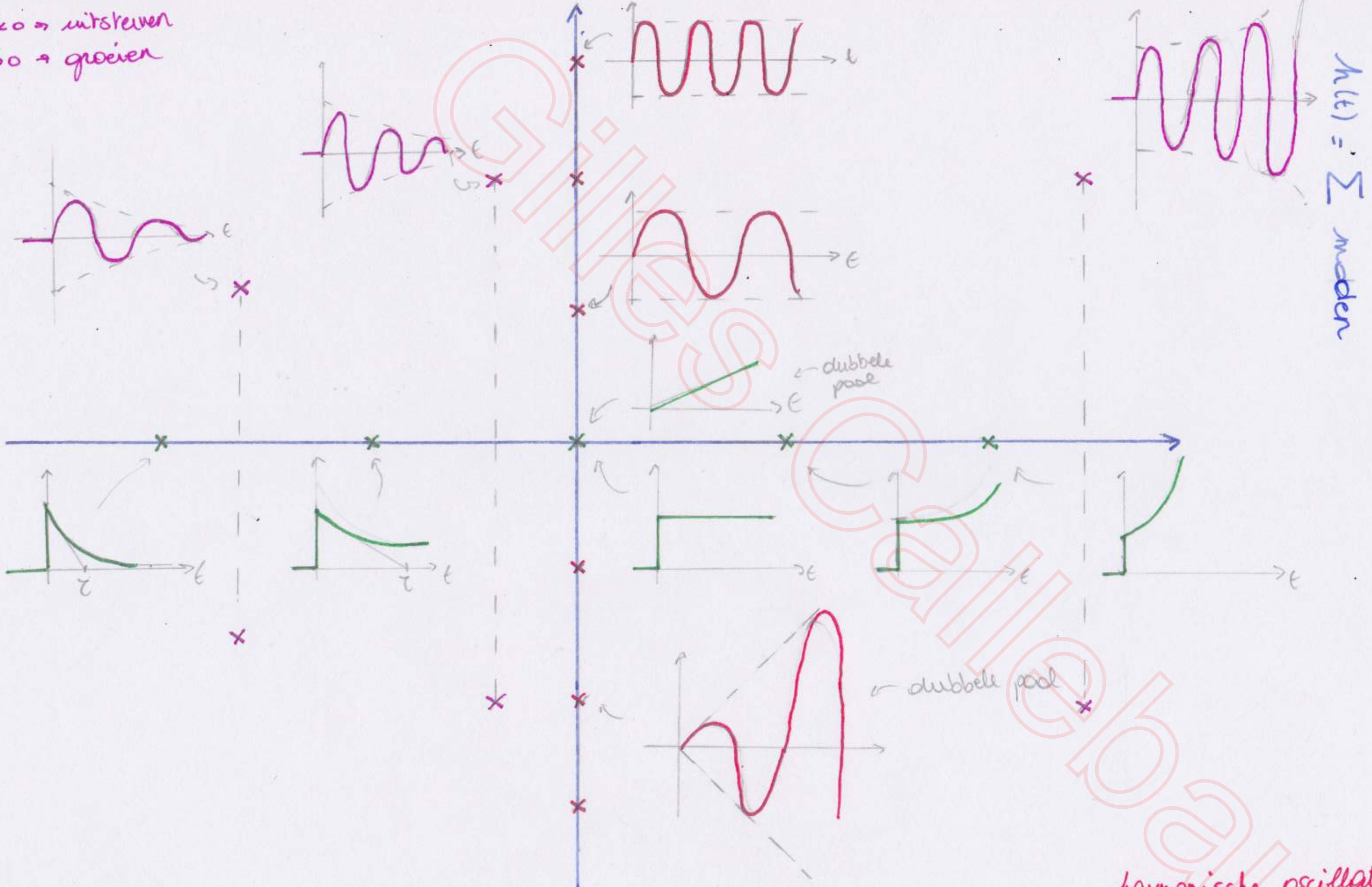
$$h(t) = \mathcal{L}^{-1}\left\{ H(s) \right\}$$

polen bepalen de vorm splitsen in partieelbreuken	inverse Laplace-transformatie
$\dfrac{A}{s-p_i}$	$A e^{p_i t}$
dubbele reële polen	$A\, t\, e^{p_i t}$
$\dfrac{As+B}{(s-p_i)(s-p_j)} = \dfrac{As+B}{(s-\alpha)^2+\omega^2}$ $p_i = \alpha + j\omega \qquad p_j = \alpha - j\omega$	$C e^{\alpha t} \sin(\omega t + \varphi)$
$\dfrac{As+B}{s^2+\omega^2}$ $p = \pm j\omega$	$C \sin(\omega t + \varphi)$
$\dfrac{As+B}{(s-p_i)^2(s-p_j)^2} = \dfrac{As+B}{((s-\alpha)^2+\omega^2)^2}$ dubbel complexe toeg. polen	$e^{\alpha t}\left[C \sin(\omega t + \varphi) + C' t \sin(\omega t + \varphi') \right]$
$\dfrac{As+B}{(s^2+\omega^2)^2}$ 2 dubbele imag. polen	$C \sin(\omega t + \varphi) + C' t \sin(\omega t + \varphi')$

oscillatorische moden

$\alpha < 0 \Rightarrow$ uitsterven

$\alpha > 0 \Rightarrow$ groeien



exp. moden met

$\tau = \frac{1}{|p_i|}$

neg. as $\rightarrow$ uitsterven

pos. as $\rightarrow$ groeien

harmonische oscillatie

cte A . α ω

De stabiliteit van een L.T.O. systeem

$h(t) \rightarrow$ uitsterven

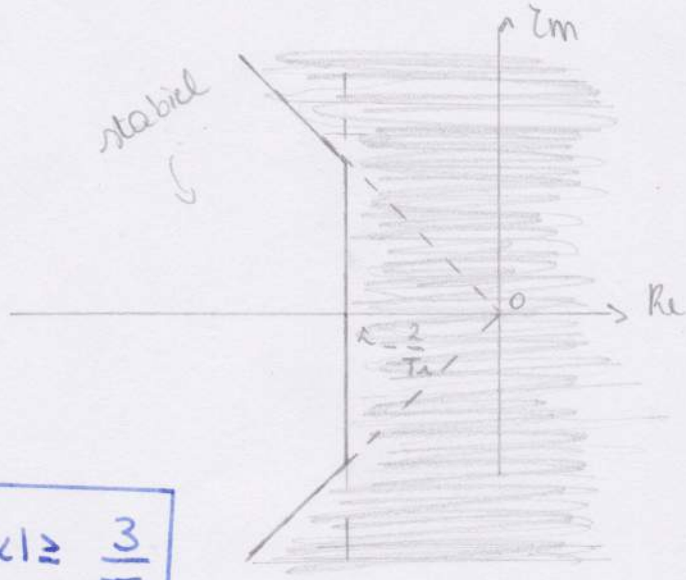
$\Rightarrow$ Opdat een L.T.O. systeem stabiel zou zijn, is het nodig en voldoende dat alle polen van zijn transferfunctie $H(s)$ i.h. linkerhalfvlak vh. PND liggen.

$\rightsquigarrow$ na responsietyd $T_r \rightarrow$ verwaarloosbaar t.o.v beginKW:

$$e^{-\frac{t}{\tau}}\Big|_{t=T_r} \leq 0{,}05$$

$$\frac{T_r}{\tau} \geq 3$$

$$\frac{1}{\tau} \geq \frac{3}{T_r} \longrightarrow \boxed{|\alpha| \geq \frac{3}{T_r}}$$



polen moeten links liggen vd verticale rechte $-\frac{3}{T_r}$

$\rightsquigarrow$ na halve periode gedempt

$\pi \approx 3$

$\hookrightarrow \quad \frac{1}{\tau} = \frac{6}{T} \longrightarrow \boxed{|\alpha| \geq \omega}$

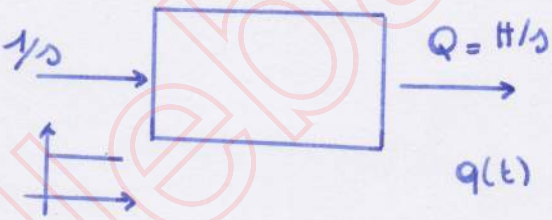
alle complexen polen tussen de bissectrices vh. tweede en 3de kwadrant (45°)

De stapresponsie v.e. $\mathcal{L}$.T.O. systeem

Algemeen

$\mathcal{L}\{\mathcal{I}^1(t)\} = \frac{1}{s}$

$$q(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$$



eerste - orde systeem

$$H(s) = \frac{1}{1+\tau s} \longrightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(1+\tau s)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+\frac{1}{\tau}}\right\}$$

$A = 1$, $B = -1$

$$\boxed{q(t) = \left[1 - e^{-t/\tau}\right]\mathcal{I}^1(t)}$$

tweede - orde systeem

$$H(s) = \frac{\omega_n^2}{s^2 + 2z\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+z\omega_n)^2 + \omega_n^2(1-z^2)}$$

$$s_{1,2} = -z\omega_n \pm \sqrt{z^2\omega_n^2 - \omega_n^2} = -z\omega_n \pm \omega_n\sqrt{z^2-1}$$

$$\longrightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s\left[(s+z\omega_n)^2 + \omega_n^2(1-z^2)\right]}\right\}$$

s = 2 versch. wortels
$s = 0$
s = 2 complexe wortels
s = 2 reële gelijke wortels

$z = 1$, kritische demping ← 2 reële wortels

$$\frac{H(s)}{s} = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{\overset{1}{A}}{s} + \frac{\overset{-1}{B}}{s+\omega_n} + \frac{\overset{-\omega_n}{C}}{(s+\omega_n)^2}$$

$$\boxed{q(t) = 1 - e^{-\omega_n t}(1+\omega_n t)}$$

$T_r = \frac{4,75}{\omega_n}$

$z > 1$, kruipende demping ← 2 reële wortels (verschillend)

← overdempt

$$s_{1,2} = -z\omega_n \pm \underbrace{\sqrt{z^2-1}}_{>0}\,\omega_n$$

$$\nearrow s_1 = -\omega_n \cdot \left[z - \underbrace{\sqrt{z^2-1}}_{>0}\right] = -\frac{1}{\tau_1}$$

$$\searrow s_2 = -\omega_n \left[\underbrace{z + \sqrt{z^2+1}}_{>0}\right] = -\frac{1}{\tau_2}$$

$$\frac{H(s)}{s} = \frac{A}{s} + \frac{B}{s+\frac{1}{\tau_1}} + \frac{C}{s+\frac{1}{\tau_2}}$$

$$x.\, s\Big|_{s=0} : \quad A = \frac{\omega_n^2}{\frac{1}{\tau_1}\cdot\frac{1}{\tau_2}} = 1$$

$$x.\left(s+\frac{1}{\tau_1}\right)\Big|_{s=-\frac{1}{\tau_1}} : \quad B = \frac{\omega_n^2}{-\frac{1}{\tau_1}\left(-\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)} = \frac{\omega_n^2}{-\frac{1}{\tau_1}\left(\frac{-\tau_2+\tau_1}{\underset{= \omega_n^2}{\tau_1\tau_2}}\right)}$$

$$C = \frac{\tau_2}{\tau_1-\tau_2}$$

$z < 1$, oscillatorische demping ← 2 complex toegevoegde wortels

$$\frac{H(s)}{s} = \frac{\omega_n^2}{s^2 + 2z\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + 2z\omega_n s + \omega_n^2} = \frac{As^2 + 2z\omega_n s A + A\omega_n^2 + Bs^2 + Cs}{s \cdot (s^2 + 2z\omega_n s + \omega_n^2)}$$

$$= \frac{(A+B)s^2 + (2z\omega_n A + C)s + A\omega_n^2}{s(s^2 + 2z\omega_n s + \omega_n^2)}$$

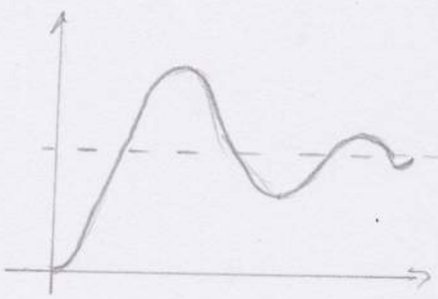
$$\longrightarrow \begin{cases} A\omega_n^2 = \omega_n^2 \\ 2z\omega_n A + C = 0 \\ A + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ C = -2z\omega_n \\ B = -1 \end{cases}$$

$$\frac{H(s)}{s} = \frac{1}{s} - \frac{s + 2z\omega_n}{s^2 + 2z\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{(s + z\omega_n) + z\omega_n}{(s + z\omega_n)^2 + (\underbrace{\omega_n\sqrt{1-z^2}}_{\omega_p})^2} \cdot \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}$$

ω_p = eigenpulsatie

$$\boxed{q(t) = 1 - e^{-z\omega_n t} \cdot \frac{1}{\sqrt{1-z^2}} \cdot \sin(\underbrace{\omega_n\sqrt{1-z^2}}_{\omega_p}\, t + \Phi)}$$

$\text{tg}\,\Phi = \dfrac{\omega}{k}$ voor de vorm $\dfrac{s+k}{s^2+\omega^2}$



$$\boxed{\text{tg}\,\Phi = \frac{1-z^2}{z}}$$

zie tabel p.84

$$\frac{H(s)}{s} = \frac{1}{s} - \frac{\tau_1}{\tau_1 - \tau_2} \cdot \frac{1}{s + \frac{1}{\tau_1}} + \frac{\tau_2}{\tau_1 - \tau_2} \cdot \frac{1}{s + \frac{1}{\tau_1}}$$

$$\boxed{q(t) = 1 - \frac{\tau_1}{\tau_1 - \tau_2} \cdot e^{-t/\tau_1} + \frac{\tau_2}{\tau_1 - \tau_2} e^{-t/\tau_2}}$$

$$T_r = 3\tau_1 = 3\,\frac{z + \sqrt{z^2 - 1}}{\omega_n}$$

$$\left.\begin{array}{l} z \text{ groot} \rightarrow \tau_1 \rightarrow \frac{2z}{\omega_n} \\ \tau_2 \rightarrow 0 \end{array}\right\} \rightarrow \text{lijken op eerste orde}$$

De harmonische responsie v.e. L.T.O. systeem

$$\begin{cases} \overline{Y} = H(j\omega)\,\overline{U} \\ H(j\omega) = H(s)\Big|_{s=j\omega} \end{cases} \rightarrow \quad H(j\omega) = k\,\frac{(j\omega - z_1)\cdots(j\omega - z_m)}{(j\omega - p_1)\cdots(j\omega - p_n)}$$

• reële nul

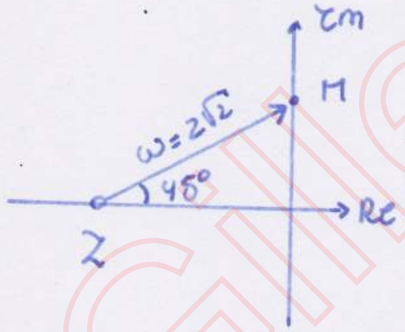
$|z|$

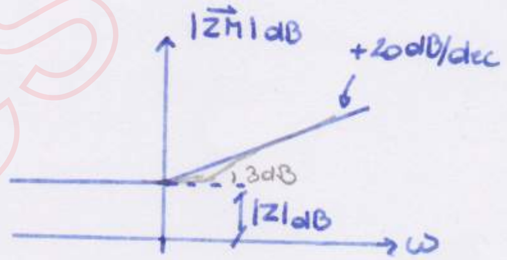
$0 \longrightarrow z\sqrt{2} \longrightarrow |Z| \sim \omega$

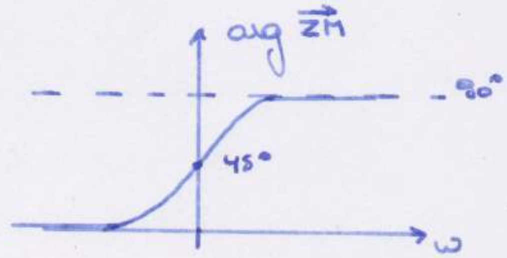
arg

bijdrage reële nul

$0° \longrightarrow 45° \longrightarrow 90°$

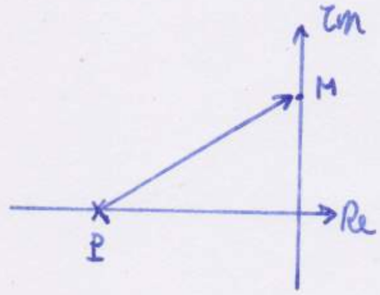


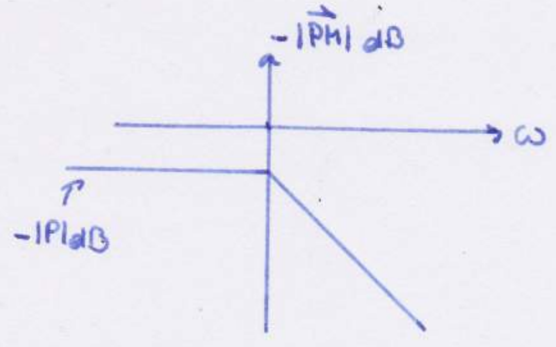


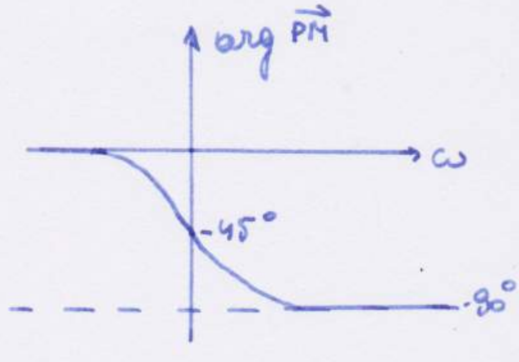


• reële pool

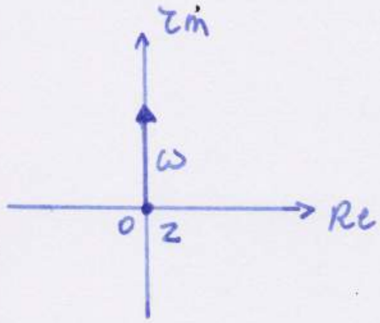
↳ gelijkaardige vector als reële nul ⇒ noemer ⇒ min-teken

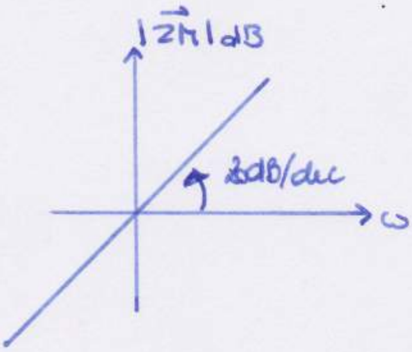


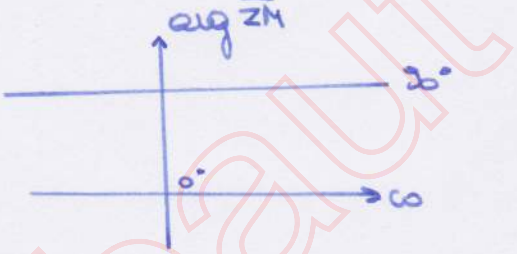




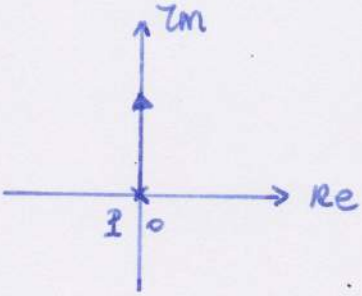
• nul i.d. oorsprong ← differentiatiefactor

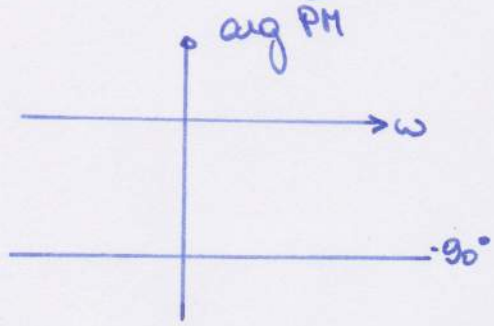






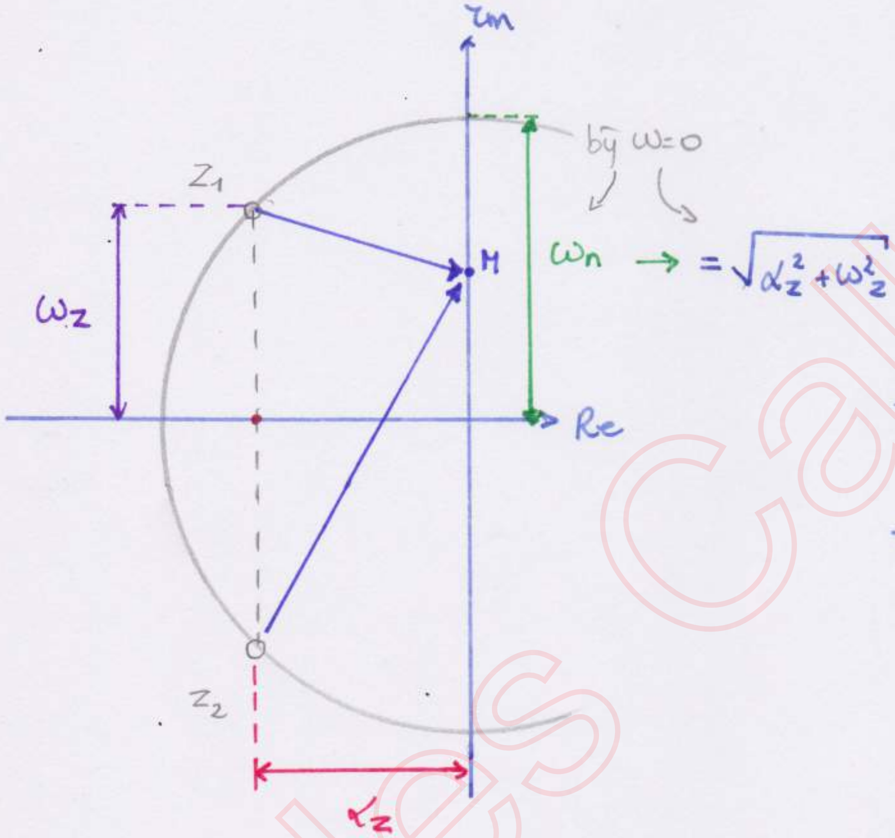
• pool i.d. oorsprong ← integratiefactor

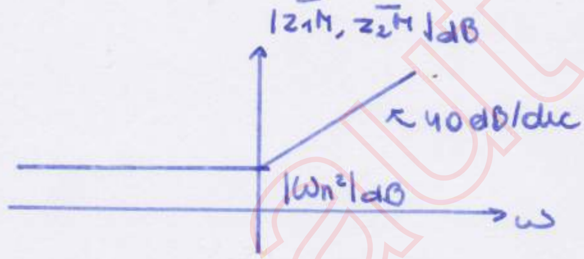


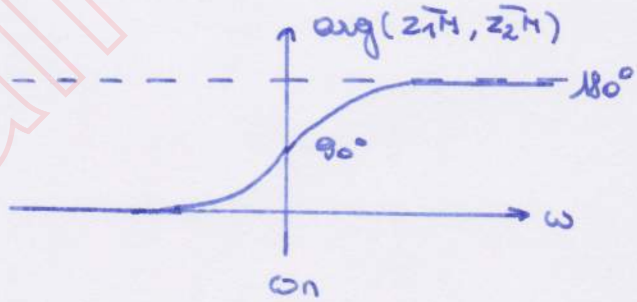


- een paar complex toegevoegde nullen

$z_1 = \alpha_z \pm j\omega_z$



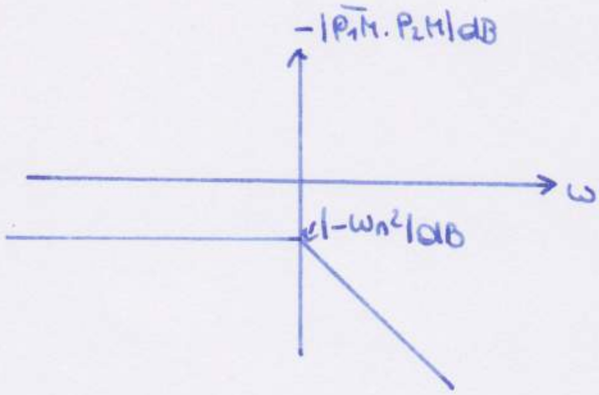


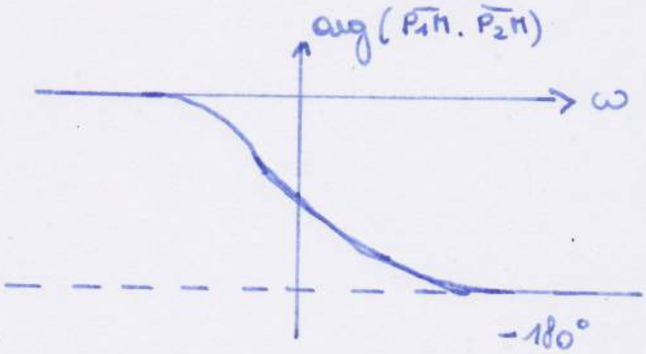


- een paar complex toegevoegde polen

$P_1 = \alpha_p \pm j\omega_p$

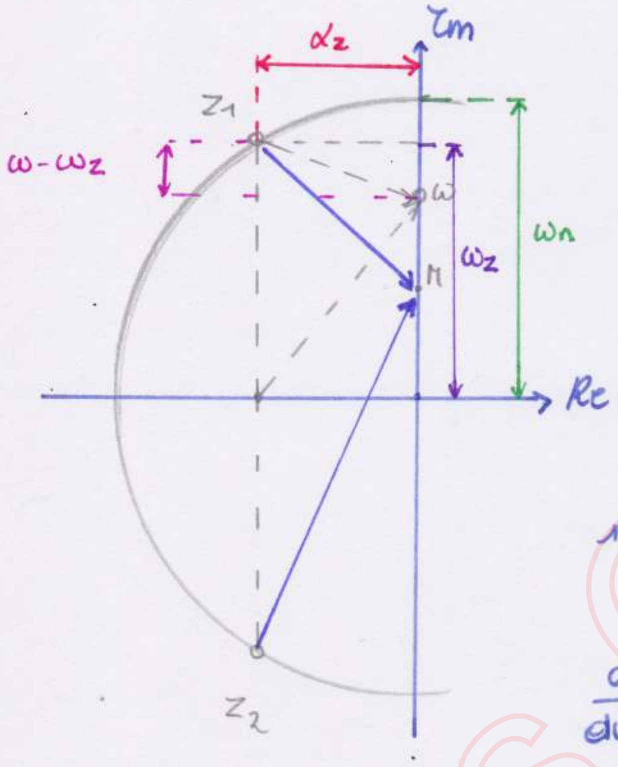
gelijkaardige vectoren
(zie hierboven)





bijdrage tot amplitudekar.

↳ max: nullen } tussen bissectrices vh
↳ min: polen } 2d & 3de kwadrant



$$|\overrightarrow{Z_1M}| = \sqrt{(\omega_z-\omega)^2+\alpha_z^2}$$

$$|\overrightarrow{Z_2M}| = \sqrt{(\omega_z+\omega)^2+\alpha_z^2}$$

minimum bepalen:

$$\min\left\{ |\overrightarrow{Z_1M}|.|\overrightarrow{Z_2M}| \right\}$$

$$\frac{d}{d\omega^2}\left(2(\omega_z^2+\omega^2+\alpha_z^2)-4\omega_z^2\right)=0$$

$$\omega^2 = \omega_z^2-\alpha_z^2$$

$$\rightarrow \boxed{\omega_{Res} = \sqrt{\omega_z^2-\alpha_z^2}}$$) als $\omega_z > \alpha_z$

complex toegev. nullen

$z_1 = a_z \pm j\omega_z \quad \rightarrow [j\omega-(\alpha_z+j\omega_z)][j\omega-(\alpha_z-j\omega_z)]$

$$= (j\omega)^2 + 2z\omega_n j\omega + \omega_n^2$$

waarbij $\begin{cases} \omega_n = \sqrt{\alpha_z^2+\omega_z^2} \\ z = \dfrac{|\alpha_z|}{\omega_n} \end{cases}$

Hoofdstuk 5

De Fourieranalyse

+

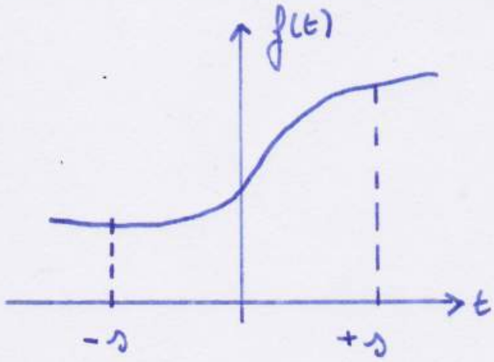
Appendix A

Gilles Callebaut

Hoofdstuk 5

De Fourieranalyse

De Fourierreeksontwikkeling



$$f(t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} \left(b_n \cos \frac{n\pi t}{s} + a_n \sin \frac{n\pi t}{s} \right)$$

$$\begin{cases} b_0 = \dfrac{1}{s} \displaystyle\int_{-s}^{+s} f(t)\, dt \\ b_n = \dfrac{1}{s} \displaystyle\int_{-s}^{+s} f(t) \cos \dfrac{n\pi t}{s}\, dt \\ a_n = \dfrac{1}{s} \displaystyle\int_{-s}^{+s} f(t) \sin \dfrac{n\pi t}{s}\, dt \end{cases}$$

som sinus & cosinus herleiden :

- $$f(t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi t}{s} + \theta_n \right)$$

- $$f(t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} g_n \cos\left(\frac{n\pi t}{s} + \theta'_n \right)$$

$$\text{met} \begin{cases} g_n = a_n^2 + b_n^2 \\ \theta_n = \mathrm{Bgtg}\, \dfrac{b_n}{a_n} \\ \theta'_n = -\mathrm{Bgtg}\, \dfrac{a_n}{b_n} \end{cases}$$

voor periodieke functie → interval $\left(-\frac{T}{2}, +\frac{T}{2}\right)$

Fourieranalyse van een periodieke tijdsfunctie

$\rightarrow s = \frac{T}{2}$ met $\omega = \frac{2\pi}{T}$

$$f(t) = \frac{b_0}{2} + \sum_{h=1}^{\infty} (b_h \cos h\omega t + a_h \sin h\omega t)$$

$\frac{b_0}{2}$ = gemiddelde waarde vd periodieke f¯ie

met
$$\begin{cases} b_0 = \frac{2}{T} \int_{-\frac{T}{2}}^{+T/2} f(t)\, dt \\ b_h = \frac{2}{T} \int_{-\frac{T}{2}}^{+T/2} f(t) \cos h\omega t\, dt \\ a_h = \frac{2}{T} \int_{-\frac{T}{2}}^{+T/2} f(t) \sin h\omega t\, dt \end{cases}$$

- even functie $(f(t) \equiv f(-t))$

  $\rightarrow a_h = 0$ voor alle h-waarden $\rightarrow$ enkel cos termen

- oneven functie $(f(t) \equiv -f(-t))$

  $\rightarrow b_0 = 0$ & $b_h = 0$ voor alle h-waarden $\rightarrow$ enkel sin termen

- halve-golfsymm. $(f(t) \equiv -f(t+\frac{T}{2}))$ of $(f(t) \equiv f(t+\frac{T}{2}))$

  enkel oneven harm. | enkel even harm.

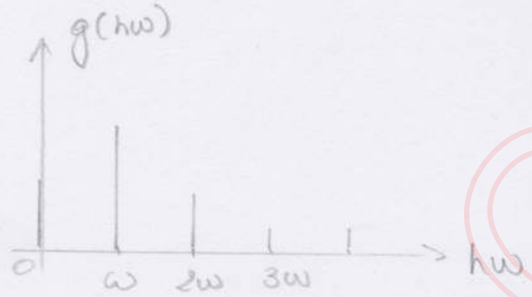
Het spectrum van een periodieke functie

$$\Rightarrow \quad f(t) = \frac{b_0}{2} + \sum_{h=1}^{\infty} g_h \cos(h\omega t + \theta'_h)$$

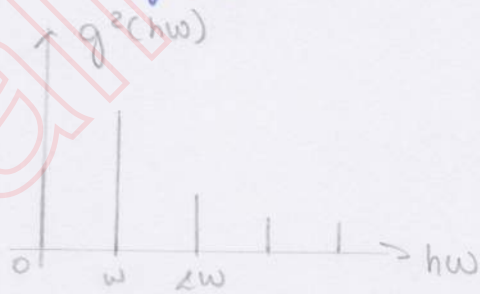
amplitude bij $\omega = 0$

$g_h = \sqrt{a_h^2 + b_h^2}$

spectrum



energiespectrum



De effectieve waarde van een periodieke tijdsfunctie

$$\Rightarrow \quad v(t) = V_0 + \sum_{h=1}^{\infty} V_h \sqrt{2} \cos(h\omega t + \beta_h)$$

gelijkspanningscomponente

V_h: effectieve waarde met freq. $h\omega$

$$\rightarrow \quad v^2(t) = V_0^2 + V_0 \sum_{h=1}^{\infty} V_h \sqrt{2} \cos(h\omega t + \beta_h) + V_0 \sum_{k=1}^{\infty} V_k \sqrt{2} \cos(k\omega t + \beta_k)$$

$$+ \left[\sum_{h=1}^{\infty} V_h \sqrt{2} \cos(h\omega t + \beta_h)\right]\left[\sum_{k=1}^{\infty} V_k \sqrt{2} \cos(k\omega t + \beta_k)\right]$$

$$\int_T v^2(t) = V_0^2 T + V_0 \sum_{h=1}^{\infty} V_h \sqrt{2} \int_T \cos(h\omega t + \beta_h)\,dt +$$

$$V_0 \sum_{k=1}^{\infty} V_k \sqrt{2} \int_T \cos(k\omega t + \beta_k)\,dt +$$

$$2 \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} V_h V_k \int_T \cos(h\omega t + \beta_h)\cos(k\omega t + \beta_k)\,dt$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

$$= V_0^2 T + 2 \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} V_h V_k \int_T \frac{1}{2}\Big[\underbrace{\cos\big((h+k)\omega t + \beta_h + \beta_k\big)}_{=0\,*}$$

$$+ \underbrace{\cos\big((h-k)\omega t + \beta_h - \beta_k\big)}_{=0\,*\text{ behalve als } h=k}\Big]\,dt$$

$*$ $\int_T \cos n\omega t\,dt = 0$ voor $n = 1, 2, 3, \ldots$

T voor $n = 0$

$$\boxed{\int_T v^2(t) = V_0^2 T + \sum_{h=1}^{\infty} V_h^2 T}$$

Het vermogen geleverd door een periodieke spanning

$$p(t) = v(t) \cdot i(t) \qquad \& \qquad P = \frac{1}{T} \int_T p(t)\, dt$$

$$P = \frac{1}{T} \left[V_0 I_0 T + I_0 \sum_{h=1}^{\infty} V_h \sqrt{2} \int_T \cos(h\omega t + \beta_h)\, dt \right.$$

$$\left. + V_0 \sum_{k=1}^{\infty} I_k \sqrt{2} \int_T \cos(k\omega t + \alpha_k)\, dt \right]$$

$$+ 2 \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} V_h I_h \int_T \cos(h\omega t + \beta_h) \cos(k\omega t + \alpha_k)\, dt$$

$$= V_0 I_0 + \frac{1}{T} \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} V_h I_k \int_T \cos\left((h+k)\omega t + \beta_h + \alpha_k\right)$$

$$+ \cos\left((h-k)\omega t + \beta_h - \alpha_h\right) dt$$

$$\boxed{P = V_0 I_0 + \sum_{h=1}^{\infty} V_h I_h \cos \phi_h \qquad \phi = \beta_h - \alpha_h}$$

De exponentiële vorm van de Fourierreeks ve per. fië

$$f(t) = \frac{b_0}{2} + \sum_{k=1}^{\infty} \left(b_k \underbrace{\cos k\omega t}_{\frac{e^{jk\omega t} + e^{-jk\omega t}}{2}} + a_k \underbrace{\sin k\omega t}_{\frac{e^{jk\omega t} - e^{-jk\omega t}}{2j}} \right)$$

$$f(t) = \frac{b_0}{2} + \sum_{k=1}^{\infty} \left(\frac{b_k - j a_k}{2} e^{jk\omega t} + \frac{b_k + j a_k}{2} e^{-jk\omega t} \right)$$

$$\frac{b_k - j a_k}{2} = \frac{1}{T} \int_{-T/2}^{+T/2} f(t)\, e^{-jk\omega t} dt \qquad (1)$$

$$\frac{b_k + j a_k}{2} = \frac{1}{T} \int_{-T/2}^{+T/2} f(t)\, e^{+jk\omega t} dt \qquad (2)$$

$\Longrightarrow$

$$c_h = \frac{1}{T} \int_{-T/2}^{+T/2} f(t)\, e^{-jh\omega t} dt$$

$\rightarrow$ voor $h = k \rightarrow (1)$

$\rightarrow$ voor $h = 0 \rightarrow c_h = \frac{b_0}{2}$

$\rightarrow$ voor $h = -k \rightarrow (2)$

$$\Longrightarrow \quad \boxed{f(t) = \sum_{h=-\infty}^{\infty} c_h\, e^{jh\omega t}}$$

Responsie van een L.T.O. systeem op een periodiek signaal

- periodiek ingangsignaal → goniometr. Fourierreeks

$$u(t) = \sum_{h=0}^{\infty} U_h \cos(h\omega t + \alpha_h)$$

waarbij $\overline{Y}_h = H(j\omega)\,\overline{U}_h$

$Y_h e^{j\beta_h}$ $\quad$ $U_h e^{j\alpha_h}$

$$y(t) = \sum_{h=0}^{\infty} Y_h \cos(h\omega t + \beta_h)$$

- per. ingangsign. → exponentiële Fourierreeks

$$u(t) = \sum_{h=-\infty}^{+\infty} \tilde{u}_h(t) = \sum_{h=-\infty}^{+\infty} u_h e^{jh\omega t}$$

waarbij: $\tilde{y}_h(t) = H(j\omega)\, u_h e^{jh\omega t}$

$$y(t) = \sum_{h=-\infty}^{+\infty} \tilde{y}_h(t)$$

De integraal van Fourier

De exponentiële vorm

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\omega t} \longrightarrow f(t) = \sum_{n=-\infty}^{+\infty} \frac{\omega}{2\pi} \int_{-s}^{+s} f(\lambda)\, e^{-jn\omega\lambda} d\lambda \, e^{jn\omega t}$$

met $c_n = \frac{1}{2s} \int_{-s}^{+s} f(\lambda)\, e^{-jn\omega\lambda} d\lambda$

Als $s \to \infty$; $\omega \quad -\infty \to +\infty$

Stellen we $n\omega = z$ dan $\omega = \Delta z$ $\qquad s \to \infty$ dan z = continue variabele & $\Delta z = dz$

$$\Rightarrow f(t) = \frac{1}{2\pi} \lim_{s \to \infty} \sum_{n=-\infty}^{+\infty} \left[\int_{-s}^{+s} f(\lambda)\, e^{-jz\lambda} d\lambda \right] e^{jzt} \Delta z$$

$\lambda \to t$

$z \to \omega$

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t} dt \right] e^{+j\omega t} d\omega}$$

$\Phi(j\omega)$ zie volgende pagina

De Fouriertransformatie

Definitie

$$\Phi(j\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

fouriergetransformeerde

$$\longrightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(j\omega)\, e^{+j\omega t}\, d\omega$$

$$\Phi(j\omega) = \mathcal{F}\{f(t)\}$$

waarbij : $|\Phi(j\omega)|$ (amplitude)-spectrum

$|\Phi(j\omega)|^2$ energiespectrum

Belangrijke Fouriergetransformeerden

- Als $f(t) = 0$ voor $t < 0$ $\rightarrow$ $\Phi(j\omega) = F(s)\Big|_{s=j\omega}$

- $\mathcal{F}\{\cos\omega_0 t\}$; $\mathcal{F}\{\sin\omega_0 t\}$

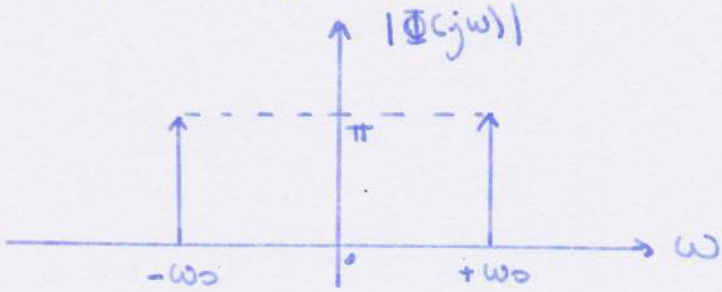
$$\begin{matrix}\cos\omega_0 t \\ \sin\omega_0 t\end{matrix} = \frac{1}{2j}\left[e^{j\omega_0 t} \pm e^{-j\omega_0 t}\right]$$

$\downarrow$ dirac : $\int_{-\infty}^{+\infty} f(t) . \delta(t-t_0)\, dt = f(t_0)$

$$= \frac{1}{2j}\left[\int_{-\infty}^{+\infty} e^{+j\omega t}\, \delta(\omega-\omega_0)\, d\omega \pm \int_{-\infty}^{+\infty} e^{+j\omega t}\, \delta(\omega+\omega_0)\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{-j\,\pi\left(\delta(\omega-\omega_0) \pm \delta(\omega+\omega_0)\right)}\, . \, e^{j\omega t}\, d\omega$$

$\mathcal{F}\{\cos\omega_0 t\} = \pi\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$ $\qquad$ $\mathcal{F}\{\sin\omega_0 t\} = j\pi\left(\delta(\omega+\omega_0) - \delta(\omega+\omega_0)\right)$



$\leftarrow$ zelfde amplitude-spectrum voor sin & cos

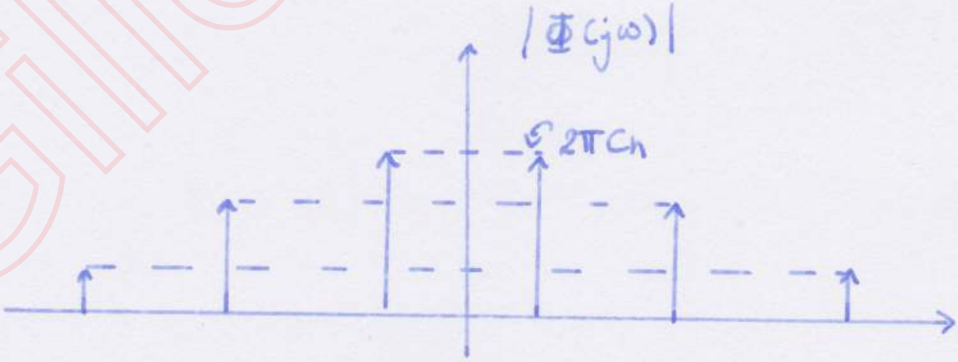
- Fouriergetransformeerde van periodische fie

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

$$\longrightarrow \quad = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} c_n \, \delta(\omega - n\omega_0) \, e^{j\omega t} \, d\omega$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{\left[\sum_{n=-\infty}^{+\infty} 2\pi c_n \, \delta(\omega - n\omega_0) \right]} e^{j\omega t} \, d\omega$$

$\Phi(j\omega)$ / $\mathcal{F}\{f(t)\}$



Eigenschappen van de Fouriertransformatie

- lineariteit

- $\Phi(j\omega) = \Phi^*(-j\omega)$ want $\bar{A} = a + bj = |A| e^{j\theta}$

  $\bar{A}^* = a - bj = |A| e^{-j\theta}$

  daaruit volgt:

$$|\Phi(j\omega)| = |\Phi(-j\omega)|$$

$$\arg \Phi(j\omega) = -\arg \Phi(-j\omega)$$

→ spectrum is dus symm. t.o.v. de $|\Phi|$-as

- Als $f(t)$ even → $f(-t) = f(t)$ → $\Phi(j\omega) = \Phi^*(j\omega)$

  ↳ dan is Φ een reële functie

  Als $f(t)$ oneven → $f(-t) = -f(t)$ → $\Phi(j\omega) = -\Phi^*(j\omega)$

  ↳ dan is Φ een zuiver imaginaire functie

- Symmetrie: $\Phi(jt) = \mathcal{F}^{-1}\{2\pi f(-\omega)\}$

- verschuiving tijdsschaal: $\Phi(j\omega)\, e^{-j\omega t_0} = \mathcal{F}\{f(t-t_0)\}$

- verandering tijdsschaal: $\frac{1}{a}\Phi\left(\frac{j\omega}{a}\right) = \mathcal{F}\{f(at)\}$

- verschuiving freq. domein: $\Phi(j(\omega-\omega_0)) = \mathcal{F}\{e^{j\omega_0 t} f(t)\}$

- convolutie in de tijd: $\Phi_1(j\omega) \cdot \Phi_2(j\omega) = \mathcal{F}\{f_1(t) * f_2(t)\}$

- convolutie i.h. freq. domein: $\frac{1}{2\pi}\left[\Phi_1(j\omega) * \Phi_2(j\omega)\right] = \mathcal{F}\{f_1(t) \cdot f_2(t)\}$

$$= \int_{-\infty}^{+\infty} \Phi_1(j\Omega)\, \Phi_2(j\omega - j\Omega)\, d\Omega$$

- Formule van Parseval

$$\underbrace{\int_{-\infty}^{+\infty} |f(t)|^2\, dt}_{\text{gedissipeerd energie over } 1\,\Omega} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{|\Phi(j\omega)|^2}_{\text{energiedichtheid/bandbreedte = spectrale energiedichth.}}\, d\omega$$

- Sommatie-formule van Poisson

$f(t)$ willekeurig $\rightarrow \sum_{n=-\infty}^{+\infty} f(nT) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} \Phi(jn\omega_s)$ met $\omega_s = \frac{2\pi}{T}$

$f(t) = 0$ voor $t < 0$ $\rightarrow \sum_{n=0}^{+\infty} f(nT) = \underbrace{\frac{f(0^+)}{2}}_{\text{correctieterm.}} + \frac{1}{T}\sum_{n=-\infty}^{+\infty} \Phi(jn\omega_s)$

De gemiddelde waarde van een analoog signaal

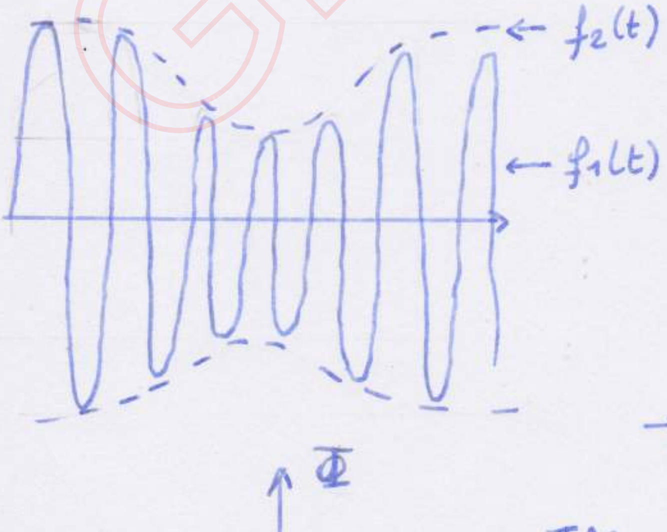
$$V_{gem} = \frac{1}{T}\int_0^T v(t)\,dt \quad \& \quad V(j\omega) = \int_0^T v(t)\, e^{-j\omega t}\,dt$$

$$\Rightarrow \quad V_{gem} = \frac{V(j0)}{T}$$

De effectieve waarde van een analoog signaal

$$V_{eff}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt \xrightarrow[\text{Parseval}]{} V_{eff}^2 = \frac{1}{2\pi T}\int_{-\infty}^{+\infty} |V(j\omega)|^2\,d\omega$$

Het spectrum van een amplitude-gemoduleerd signaal



met $\begin{cases} f_1(t) = A_c \cos\omega_c t \\ f_2(t) = 1 + m\cos\omega_m t \end{cases}$

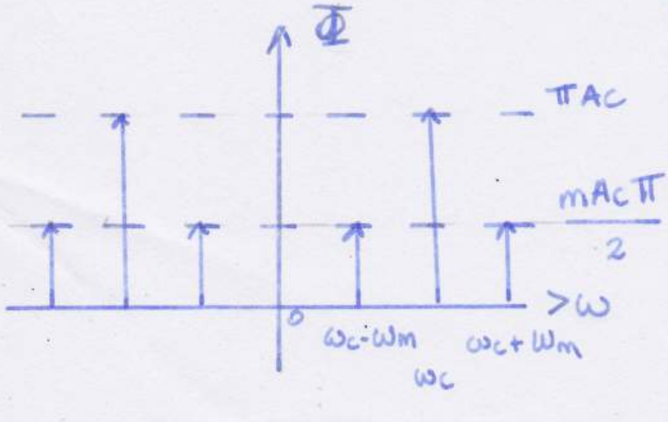
m: modulatiediepte

$\omega_c \gg \omega_m$

$\rightarrow \quad f_1(t) \cdot f_2(t)$

$\rightarrow \quad \mathcal{F}\{f_1(t) \cdot f_2(t)\}$

$$= \frac{1}{2\pi}\left[\Phi_1(j\omega) * \Phi_2(j\omega)\right]$$



De responsie van een L.T.O. systeem

$$y(t) = \int_0^t u(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$\Rightarrow \underbrace{Y(j\omega)}_{=\mathcal{F}\{y(t)\}} = \underbrace{H(j\omega)}_{=\mathcal{F}\{h(t)\}}\, \underbrace{U(j\omega)}_{=\mathcal{F}\{u(t)\}}$$

$h(t) = 0$ voor $t < 0$

$$\Rightarrow \underbrace{H(j\omega)}_{\text{Fourier-getransf.}} = \underbrace{H(s)}_{\text{Laplace-getransf.}}\Big|_{s=j\omega}$$

Appendix A:

Stabiliteitscriterium van Routh-Hurwitz

stabiel als karakt. verg. geen pos. reële wortels

↳ alle coëff. hetzelfde teken.

R-H tabel

$$
\begin{array}{c|cccc}
s^n & 1 & b_{n-2} & b_{n-4} & \dots \\
s^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\
s^{n-2} & c_1 & c_2 & c_3 & \dots \\
s^{n-3} & d_1 & d_2 & \dots & \\
\dots & & & & \\
s^0 & m_1 & & &
\end{array}
$$

↑

tekenomwisseling

=

wortels met pos. Re. deel

met

$$c_1 = \frac{\begin{vmatrix} 1 & b_{n-2} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$$

$$c_2 = \frac{\begin{vmatrix} 1 & b_{n-4} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}} \quad \dots$$

$$d_1 = \frac{\begin{vmatrix} b_{n-1} & b_{n-3} \\ c_1 & c_2 \end{vmatrix}}{c_1} \quad \dots$$

Problemen

- eerste element rij $= 0$ → vervangen door ε zeer klein getal
- nul rij → rij erboven afleiden → invullen ipv. nulrij.

Deel 2:

Tijdsdiscrete

Systemen

Gilles Callebaut

Hoofdstuk 1

Tijddiscrete systemen

Gilles Callebaut

lineaire tijdsonafhankelijke discrete systemen

Definitie

$$u(nT) \longrightarrow \boxed{\text{L.T.D.}} \longrightarrow y(nT)$$

- lineariteit: superpositie
- Tijdsonafh.: $u(nT) \rightarrow y(nT)$

  $u(nT-kT) \rightarrow y(nT-kT)$
- causaliteit: enkel afhankelijk van de ingang op dat moment

Het wiskundig model

de differentievergelijking

N^e orde systeem

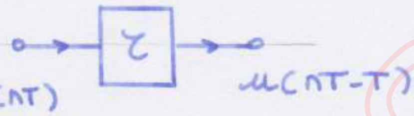
$$y(nT) = \sum_{i=0}^{N} a_i \, u(nT-iT) - \sum_{i=1}^{N} b_i \, y(nT-iT)$$

als $b_i = 0$

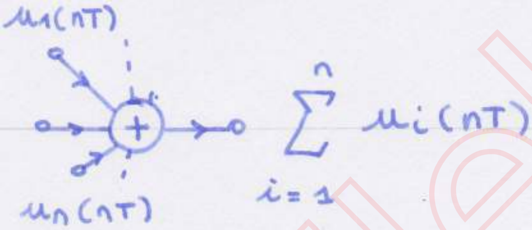
→ nt recursief systeem

Basisbouwstenen voor een LTD systeem

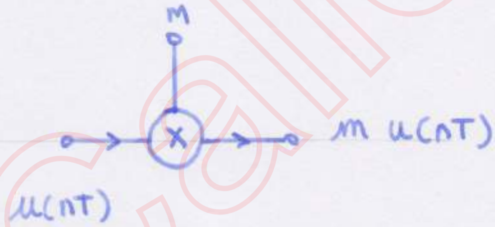
- eenheidsvertragingselement



- de opteller



- de vermenigvuldiger



Hoofdstuk 2

Studie van het gedrag van Lineaire Tijdsonafh. Discrete Systemen

Gilles Callebaut

Studie van LTD systemen in tijdsdomein

De impulsresponsie

$$\delta(nT) = \begin{cases} 1 & \text{voor } n=0 \\ 0 & \text{voor } n \neq 0 \end{cases}$$

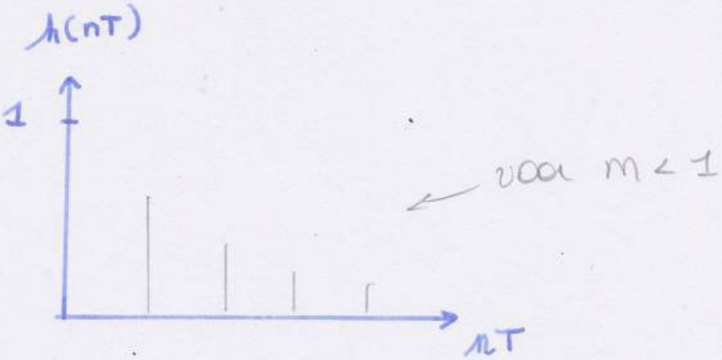
1e orde systeem:

$$y(nT) = \underbrace{x(nT)}_{=\,\delta(nT)} + m\, y(nT-T)$$

n	
0	$y(0) = 1 + m \cdot y(-T) = 1$
1	$y(T) = 0 + m\, y(0) = m$
	$\dots$
k	$y(kT) = 0 + m\, y((k-1)T) = m^k$

y voor tijdstip 0 = 0

$$\Longrightarrow \boxed{h(nT) = m^n}$$ ← impulsresponsie 1e orde systeem



De responsie op een willekeurige tijdsdiscrete ingang

$u(t) = \#$ componenten

$$u(nT) = \sum_{k=0}^{n} u(kT) \cdot \delta(nT - kT)$$

$$\downarrow$$

$$y(nT) = \sum_{k=0}^{n} u(kT) \cdot h(nT - kT)$$

convolutiesom

$$y(nT) = u(nT) * h(nT)$$

stapresponsie

$$y(nT) = \sum_{k=0}^{n} I(nT - kT)\, h(kT)$$

$$q(nT) = \sum_{k=0}^{n} m^k = 1 + m + \dots + m^n$$

$$q(nT) = \frac{1 - m^{n+1}}{1 - m}$$

De z-transformatie

Definitie

$$Z[f(nT)] = F(z) = \sum_{n=0}^{\infty} f(nT) \,.\, z^{-n}$$

De z-getransf. van enkele elementaire functies

- eenheidsimpuls

$$Z[\delta(nT)] = \sum_{n=0}^{\infty} \underbrace{\delta(nT)}_{\substack{1 \text{ voor } n=0 \\ 0 \text{ voor } n\neq 0}} \,.\, z^{-n} = 1$$

- eenheidsstap

$$Z[\Gamma(nT)] = \sum_{n=0}^{\infty} \underbrace{\Gamma^{+}(nT)}_{1 \text{ vanaf } n\geq 0} \,.\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{n}$$

$$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

- $\cos(n\pi) = (-1)^n$

$$Z[\cos(n\pi)] = \sum_{n=0}^{\infty} \underbrace{\cos(n\pi)}_{=(-1)^n} \; z^{-n}$$

$$= z^0 - z^{-1} + z^{-2} - \ldots$$

$$= \frac{1}{1+\frac{1}{z}} = \frac{z}{z+1}$$

Eigenschappen v.d. z-transformatie

- Lineariteit

- Tydverschuiving

$$Z\left[f(nT-mT)\right] = \sum_{n-m=0}^{\infty} f((n-m)T)\, z^{-n}$$

$$= \sum_{n=m}^{\infty} f(\overset{n'}{(n-m)}T)\, z^{-\overset{n'}{(n-m)}}\, z^{-m}$$

$$= z^{-m} \sum_{n'=0}^{\infty} f(n'T) . z^{-n'} = F(z) . z^{-m}$$

$$Z\left[f(nT+mT)\right] = \sum_{n+m=0}^{\infty} f(\overset{n'}{(n+m)}T) . z^{-\overset{n'}{(n+m)}} . z^{m}$$

$$= \sum_{n'=m}^{\infty} f(n'T) . z^{-n'} . z^{m}$$

$$= \left(\sum_{n'=0}^{\infty} f(n'T) . z^{-n'} - \sum_{n'=0}^{m-1} f(n'T) . z^{-n'} \right) . z^{m}$$

$$= F(z) . z^{m} + \sum_{n'=0}^{m-1} f(nT) . z^{-n} . z^{m}$$

- vermenigvuldigen met een exponentiële reeks

$$Z[a^n f(nT)] = \sum_{n=0}^{\infty} a^n f(nT) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \cdot \left(\frac{z}{a}\right)^{-n}$$

$$= F\left(\frac{z}{a}\right)$$

kan $e^{\alpha T}$ zijn kan $-a$ zijn ...

- vermenigvuldigen met de tijd

$$\frac{dF(z)}{dz} = \sum_{n=0}^{\infty} -n\, f(nT) \cdot z^{-n-1} = -z^{-1} \sum_{n=0}^{\infty} n f(nT) \cdot z^{-n}$$

$$Z[nT\, f(nT)] = \sum_{n=0}^{\infty} nT\, f(nT) \cdot z^{-n}$$

$$= -Tz \frac{dF(z)}{dz}$$

- beginwaardestelling

$$\lim_{z \to \infty} F(z) = \lim_{z \to \infty} \sum_{n=0}^{\infty} f(nT) \cdot z^{-n}$$

$$= \lim_{z \to \infty} \left[f(0) + \frac{f(T)}{z} + \dots \right] = f(0)$$

$$= \lim_{t \to 0} f(t)$$

- eindwaardestelling

$$S_n = f(0) + \frac{f(T)}{z} + \dots + \frac{f(nT)}{z^n}$$

$$S_{n-1} = f(0) + \frac{f(T)}{z} + \dots + \frac{f((n-1)T)}{z^{n-1}}$$

$$\Rightarrow S_n - \frac{1}{z} S_{n-1} = \left(1 - \frac{1}{z}\right) \left[f(0) + \dots + \frac{f((n-1)T)}{z^{n-1}} \right] + \frac{f(nT)}{z^n}$$

$$\lim_{z \to 1} \left(S_n - \frac{1}{z} S_{n-1}\right) = f(nT)$$

voor $n \to \infty \to S_n = F(z)$, $S_{n-1} = F(z)$

$$\lim_{z \to 1} \frac{z-1}{z} \cdot F(z) = f(\infty)$$

• convolutiestelling

$$F(z).G(z) = \sum_{k=0}^{\infty} f(kT)\, \underbrace{z^{-k}\, G(z)}_{\downarrow \text{ tijdsverschuiving}}$$

$$= \sum_{k=0}^{\infty} f(kT).\, Z[\, g(nT-kT)]$$

$$= \sum_{k=0}^{\infty} f(kT) \sum_{n=0}^{\infty} g(nT-kT).\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} f(kT)\, g(nT-kT) \right] .\, z^{-n}$$

$$= Z\left[f(kT).\, g(nT-kT) \right]$$

De inverse z-transformatie

splitsen in partieelbreuken

staartdeling vb.

$$F(z) = \frac{z}{z^2 - 3z + 2}$$

$$F(z) = \frac{1}{z^{n-1}} \cdot \frac{z^n}{z^2 - 3z + 2}$$

$$\begin{array}{l|l} z^n & z^2 - 3z + 2 \\ \hline & z^{n-2} + \dots \end{array}$$

$$F(z) = \left(z^{n-2} + 3z^{n-3} + \dots\right) \cdot \frac{1}{z^{n-1}}$$

$$F(z) = z^{-1} + 3z^{-2} + \dots$$

$$f(nT) = 1 . \delta(nT - T) + 3 . \delta(nT - 2T)$$

Studie v.h. gedrag v.e. LTD systeem op basis v.d. transferfunctie

$$y(nT) = \sum_{i=0}^{N} a_i \cdot u(nT - iT) - \sum_{i=1}^{N} b_i \cdot y(nT - iT)$$

$$Y(z) = \sum_{i=0}^{N} a_i \, z^{-i} \cdot U(z) - \sum_{i=1}^{N} b_i \cdot z^{-i} \cdot Y(z)$$

$$\frac{Y(z)}{U(z)} = H(z) = \frac{\sum_{i=0}^{N} a_i \, z^{N-i}}{z^N + \sum_{i=1}^{N} b_i \, z^{N-i}}$$

Het PND van H(z)

nulpunten (o)

$$H(z) = k \, \frac{(z - z_1)(z - z_2)\ldots}{(z - p_1)(z - p_2)\ldots}$$

polen (x)

Niet recursief systeem ⇒ alle polen samen i.d. oorsprong

De impulsresponsie v.e. LTD systeem - de stabiliteit

de invloed v.d. ligging v.d. polen op de impulsresponsie

enkelvoudige polen : $A \cdot \dfrac{z}{z - p_i}$ (1)

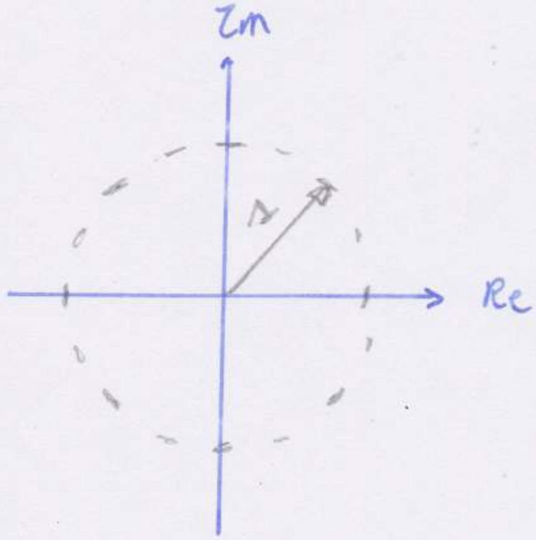
dubbele polen: $A \cdot \dfrac{z}{(z-p_i)^2}$ (2)

(1) $Z^{-1}\left[A \dfrac{z}{z-p_i}\right] = A \cdot (p_i)^n \cdot \mathcal{I}'(nT)$

als $n \to \infty$:
$= 0 \quad |p_i| < 1$
$= \text{ct.} \quad |p_i| = 1$
$= \infty \quad |p_i| > 1$

(2) $Z^{-1}\left[A \dfrac{z}{(z-p_i)^2}\right] = A \cdot \dfrac{1}{p_i T} \cdot nT \cdot (p_i)^n \cdot \mathcal{I}'(nT)$

als $n \to \infty$: $= 0 \quad |p_i| < 1$



stabiel als polen binnen/op de eenheidscirkel liggen, maar niet op bij dubbele polen.